Sequence & Series

"1729 is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." ...... S.Ramanujan

Sequence :

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$\{f(1), f(2), f(3), ...\} = \{t_1, t_2, t_3, ...\} = \{t_n\}$

Real sequence :

A sequence whose range is a subset of R is called a real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

Types of sequence :

On the basis of the number of terms there are two types of sequence.

(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences : A sequence is said to be infinite if it has infinitely many terms.

Example # 1 : Write down the sequence whose n^{th} term is $\frac{(-2)^n}{(-1)^n + 2}$

Solution : Let $t_n = \frac{(-2)^n}{(-1)^n + 2}$

put n = 1, 2, 3, 4, ... we get

$t_1 = -2$, $t_2 = \frac{4}{3}$, $t_3 = -8$, $t_4 = \frac{16}{3}$

so the sequence is $-2, , -8, \frac{16}{3}$, ...

Series :

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ...a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

e.g. (i) 1 + 2 + 3 + 4 + ... + n

(ii) 2 + 4 + 8 + 16 + ...

(iii) – 1 + 3 – 9 + 27 – ....

Progression :

The word progression refers to sequence or series – finite or infinite

Arithmetic progression (A.P.) :

A.P. is a sequence whose successive terms are obtained by adding a fixed number 'd' to the preceding terms. This fixed number 'd' is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... ,a + (n – 1) d,........

e.g. – 4, – 1, 2, 5 ....

n^{th} term of an A.P. :

Let 'a' be the first term and 'd' be the common difference of an A.P., then

$t_n = a + (n - 1) d$, where $d = t_n - t_{n-1}$

Example # 2 : Find the number of terms in the sequence 4, 7, 10, 13, ...,82.

Solution : Let a be the first term and d be the common difference

a = 4, d = 3 so 82 = 4 + (n – 1)3

$\Rightarrow$ n = 27

The sum of first n terms of an A.P. :

If a is first term and d is common difference, then sum of the first n terms of AP is

$S_n = \frac{n}{2}$ [2a + (n – 1) d]

$= \frac{n}{2}$ $[a + \ell] \equiv n t_{\left(\frac{n+1}{2}\right)}$, for n is odd. (Where ℓ is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.)

Note : For any sequence $\{t_n\}$, whose sum of first r terms is S_r, r^{th} term, $t_r = S_r - S_{r-1}$.

Example # 3 : If in an A.P., 3rd term is 18 and 7 term is 30, then find sum of its first 17 terms

Solution : Let a be the first term and d be the common difference

a + 2d = 18

a + 6d = 30

d = 3 , a = 12

$s_{17} = \frac{17}{2}$ [2 × 12 + 16 × 3] = 612

Example # 4 : Find the sum of all odd numbers between 1 and 1000 which are divisible by 3

Solution : Odd numbers between 1 and 1000 are

3, 5, 7, 9, 11, 13, ------ 993, 995, 997, 999.

Those numbers which are divisible by 3 are

3, 9, 15, 21, ------- 993, 999

They form an A.P. of which a = 3 , d = 6, ℓ = 999 $\therefore$ n = 167

$S = \frac{n}{2}$ $[a + \ell]$ = 83667

Example # 5 : The ratio between the sum of n term of two A.P.'s is 3n + 8 : 7n + 15. Then find the ratio between their 12 th term

Solution : $\frac{S_n}{S_n'} = \frac{(n/2)[2a+(n-1)d]}{(n/2)[2a'+(n-1)d']} = \frac{3n+8}{7n+15}$ or $\frac{a+\{(n-1)/2\}d}{a'+(n-1)/2d'} = \frac{3n+8}{7n+15}$ ----- (i)

we have to find $\frac{T_{12}}{T_{12}'} = \frac{a+11d}{a'+11d'}$

choosing (n – 1)/2 = 11 or n = 23 in (1),

we get $\frac{T_{12}}{T_{12}'} = \frac{a+11d}{a'+11d'} = \frac{3(23)+8}{(23)\times 7+15} = \frac{77}{176} = \frac{7}{16}$

Example # 6 : If sum of n terms of a sequence is given by $S_n = 3n^2 - 4n$, find its 50^{th} term.

Solution : Let t_n is n^{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= 3n^2 - 4n - 3(n-1)^2 + 4(n-1) = 6n - 7$

so t_{50} = 293.

Self practice problems :

(1) Which term of the sequence 2005, 2000, 1995, 1990, 1985, ... is the first negative term

(2) For an A.P. show that $t_m + t_{2n+m} = 2\, t_{m+n}$

(3) Find the maximum sum of the A.P. 40 + 38 + 36 + 34 + 32 + ...

(4) Find the sum of first 16 terms of an A.P. a_1, a_2, a_3..........
If it is known that $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 147$

Ans. (1) 403 (3) 420 (4) 392

Remarks :

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow$ 2 b = a + c & if a, b, c, d are in A.P. $\Rightarrow$ a + d = b + c.

(iii) Three numbers in A.P. can be taken as a – d, a, a + d ; four numbers in A.P. can be taken as a – 3d, a – d, a + d, a + 3d; five numbers in A.P. are a – 2d, a – d, a, a + d, a + 2d ; six terms in A.P. are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = 1/2\,(a_{n-k} + a_{n+k})$, k < n. For k = 1, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$;
For k = 2, $a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

(vii) The sum and difference of two AP's is an AP.

Example # 7 : The numbers t (t^2 + 1), $-\dfrac{t^2}{2}$ and 6 are three consecutive terms of an A.P. If t be real, then find the the next two term of A.P.

Solution : 2b = a + c $\Rightarrow$ $-t^2 = t^3 + t + 6$

or $t^3 + t^2 + t + 6 = 0$

or $(t + 2)(t^2 - t + 3) = 0$ $\therefore$ $t^2 - t + 3 \neq 0$ $\Rightarrow$ t = –2

the given numbers are – 10, – 2, 6

which are in an A.P. with d = 8. The next two numbers are 14, 22

Example # 8 : If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq$ 0, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution : As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.

Hence $\sum_{i=1}^{5} a_i = 10$.

Example # 9 : If a(b + c), b(c + a), c(a + b) are in A.P., prove that $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are also in A.P.

Solution : $\because$ a(b + c), b(c + a), c(a + b) are in A.P. $\Rightarrow$ subtract ab + bc + ca from each

– bc, – ca, – ab are in A.P.

divide by –abc

$\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

Example # 10 : If $\dfrac{a+b}{1-ab}$, b, $\dfrac{b+c}{1-bc}$ are in A.P. then prove that $\dfrac{1}{a}$, b $\dfrac{1}{c}$ are in A.P.

Solution : $\therefore$ $\dfrac{a+b}{1-ab}$, b, $\dfrac{b+c}{1-bc}$ are in A.P.

$$b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b$$

$$\frac{-a(b^2+1)}{1-ab} = \frac{c(1+b^2)}{1-bc}$$

$\Rightarrow$ –a + abc = c – abc

a + c = 2abc

divide by ac

$\dfrac{1}{c} + \dfrac{1}{a} = 2b$ $\Rightarrow$ $\dfrac{1}{a}$, b, $\dfrac{1}{c}$ are in A.P.

Arithmetic mean (mean or average) (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n numbers $a_1, a_2, ..., a_n$ is; $A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$.

n-Arithmetic means between two numbers :

If a, b are any two given numbers & a, $A_1, A_2, ..., A_n$, b are in A.P., then $A_1, A_2, ... A_n$ are the n A.M.'s between a & b.

$A_1 = a + \frac{b - a}{n + 1}$, $A_2 = a + \frac{2 (b - a)}{n + 1}$,......, $A_n = a + \frac{n (b - a)}{n + 1}$

Note : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$, where A is the single A.M. between a & b   i.e. $A = \frac{a + b}{2}$

Example # 11 : If a, b, c,d,e, f are A. M's between 2 and 12, then find a + b + c + d + e + f.

Solution : Sum of A.M.^{s} = 6 single A.M. = $\frac{6(2 + 12)}{2}$ = 42

Example # 12 : Insert 10 A.M. between 3 and 80.

Solution : Here 3 is the first term and 80 is the 12^{th} term of A.P. so 80 = 3 + (11)d

$\Rightarrow$ d = 7

so the series is 3, 10, 17, 24, ..., 73, 80

$\therefore$ required means are 10, 17, 24, ..., 73.

Self practice problems :

(5) There are n A.M.'s between 3 and 29 such that 6th mean : (n – 1)th mean : : 3 : 5 then find the value of n.

(6) For what value of n, $\frac{a^{n+3} + b^{n+3}}{a^{n+2} + b^{n+2}}$, $a \neq b$ is the A.M. of a and b.

Ans. (5) n = 12 (6) n = –2

Geometric progression (G.P.) :

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by that which immediately preceeds it. Therefore a, ar, ar^2, ar^3, ar^4,...... is a G.P. with 'a' as the first term & 'r' as common ratio.

e.g. (i) 2, 4, 8, 16, ...   (ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, ...

Results : (i) n^{th} term of GP = $a\,r^{n-1}$

(ii) Sum of the first n terms of GP

$$S_n = \begin{cases} \frac{a\left(r^n - 1\right)}{r - 1} & , \quad r \neq 1 \\ na & , \quad r = 1 \end{cases}$$

(iii) Sum of an infinite terms of GP when $|r| < 1$. When $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore,

$S_\infty = \frac{a}{1 - r} \left(|r| < 1\right)$

Example # 13 : The n^{th} term of the series 3, $\sqrt{3}$, 1 , ------ is $\frac{1}{243}$, then find n

Solution : $3.\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243} \Rightarrow n = 13$

Example # 14 : The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution : Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow$ $r = \frac{r^2}{1-r}$ $\Rightarrow$ $r = \frac{1}{2}$,

Hence series is 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...∞

Example # 15 : In a G.P.., $T_2 + T_5 = 216$ and $T_4 : T_6 = 1 : 4$ and all terms are integers, then find its first term :

Solution : ar (1 + r^3) = 216 and $\frac{ar^3}{ar^5} = \frac{1}{4}$

$\Rightarrow$ $r^2 = 4 \Rightarrow r = \pm 2$

when r = 2 then 2a(9) = 216 $\Rightarrow$ a = 12

when r = – 2, then – 2a (1–8) = 216

$\therefore$ $a = \frac{216}{14} = \frac{108}{7}$, which is not an integer.

Self practice problems :

(7) Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.

(8) If x, 2y, 3z are in A.P. where the distinct numbers x, y, z are in G.P. Then find the common ratio of G.P.

(9) A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 , then find the common ratio of the progression.

(10) If continued prodcut of three number in G.P. is 216 and sum of there product in pairs is 156. Find the numbers.

Ans. (7) 2, 6, 18, 54, 162, 486 (8) $\frac{1}{3}$ (9) $\frac{S_2}{S_1}$.

(10) 2, 6, 18

Remarks :

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if a_1, a_2, a_3, a_4,......... a_{n-1} , a_n are in G.P., then $a_1a_n = a_2a_{n-1} = a_3\, a_{n-2}$ = ...

(ii) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as, $\frac{a}{r^3}$, $\frac{a}{r}$ ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non–zero quantity, the resulting sequence is also a G.P..

(v) If a_1, a_2, a_3,........ and b_1, b_2, b_3,......... are two G.P's with common ratio r_1 and r_2 respectively, then the sequence a_1b_1, a_2b_2, a_3b_3, ... is also a G.P. with common ratio $r_1\, r_2$.

(vi) If a_1, a_2, a_3,..........are in G.P. where each $a_i > 0$, then log a_1, loga_2, loga_3,..........are in A.P. and its converse is also true.

Example # 16 : Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is :

Solution : Three number in G.P. are $\frac{a}{r}$, a, ar

then $\frac{a}{r}$, 2a ar are in A.P. as given.

$\therefore$ $2(2a) = a\left(r + \frac{1}{r}\right)$

or $r^2 - 4r + 1 = 0$

or $r = 2 \pm \sqrt{3}$

or $r = 2 + \sqrt{3}$ as r > 1 for an increasing G.P.

Example # 17 : The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of this infinite series is 24. Then find its first term and common ratio :

Solution : Let a be the first term and r be the common ratio of G.P.

$\frac{a}{1-r} = 2, \frac{a^3}{1-r^3} = 24$, $-1 < r < 1$

Solving we get a = 3, $r = -\frac{1}{2}$

Example # 18 : Express $0.4\dot{2}\dot{3}$ in the form of $\frac{p}{q}$, (where p, q $\in$ I, q $\neq$ 0)

Solution : $S = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + ... \infty = \frac{4}{10} + \frac{a}{1-r} = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$

Example # 19 : Evaluate 9 + 99 + 999 + ... upto n terms.

Solution : Let S = 9 + 99 + 999 + ...upto n terms.

= [9 + 99 + 999 + ...]

= $[(10 - 1) + (10^2 - 1) + (10^3 - 1) + ... +$ upto n terms]

= $[10 + 10^2 + 10^3 + ...+ 10^n - n] = \left(\frac{10(10^n - 1)}{9} - n\right)$

Geometric means (mean proportional) (G.M.):

If a, b, c are in G.P., b is called as the G.M. of a & c.

If a and c are both positive, then $b = \sqrt{ac}$ and if a and c are both negative, then $b = -\sqrt{ac}$.

$b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0.

n-Geometric means between a, b :

If a, b are two given numbers & a, G_1, G_2,....., G_n, b are in G.P.. Then

G_1, G_2, G_3,...., G_n are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,......, $G_n = a(b/a)^{n/n+1}$

Note : The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$, where G is the single G.M. between a & b.

Example # 20 : Between 4 and 2916 are inserted odd number (2n + 1) G.M's. Then the (n + 1)th G.M. is

Solution : 4, G_1.G_2, ... G_{n+1}, ... G_{2n}, G_{2n+1}, 2916

G_{n+1} will be the middle mean of (2n +1) odd means and it will be equidistant from Ist and last term

$\therefore$ 4,G_{n+1} , 2916 will also be in G.P.

$\therefore$ $G_{n+1}^2 = 4 \times 2916 = 4 \times 9 \times 324 = 4 \times 9 \times 4 \times 81$

$G_{n+1} = 2 \times 3 \times 2 \times 9 = 108.$

Self practice problems :

(11) Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b.

(12) If a = $\underbrace{111\ ....1}_{55}$, b = 1 + 10 + 10^2 + 10^3 + 10^4 and c = 1 + 10^5 + 10^{10} + ... + 10^{50}, then prove that

(i) 'a' is a composite number (ii) a = bc.

Ans. (11) $n = -\frac{1}{2}$

Harmonic progression (H.P.)

A sequence is said to be in H.P if the reciprocals of its terms are in A.P.. If the sequence $a_1, a_2, a_3, ...,$ a_n is in H.P. then $1/a_1, 1/a_2, ..., 1/a_n$ is in A.P.

Note : (i) Here we do not have the formula for the sum of the n terms of an H.P.. For H.P. whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

(ii) If a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

(iii) If a, b, c are in A.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{a}$

(iv) If a, b, c are in G.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{b}$

Harmonic mean (H.M.):

If a, b, c are in H.P., b is called as the H.M. between a & c, then $b = \frac{2ac}{a+c}$

If a_1, a_2 , ... a_n are 'n' non-zero numbers then H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

Example # 21 : The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

Solution : Let a be the first term and d be the common difference of corresponding A.P.

a + 6d = 10

a + 11d = 25

5d = 15

d = 3, a = – 8

T_{20} = a + 19d

= – 8 + 19 × 3 = 49

20 term of H.P. = $\frac{1}{49}$

Example # 22 : Insert 4 H.M between $\frac{3}{4}$ and $\frac{3}{19}$.

Solution : Let 'd' be the common difference of corresponding A.P..

so $d = \frac{\frac{19}{3} - \frac{4}{3}}{5} = 1.$

$\therefore \quad \frac{1}{H_1} = \frac{4}{3} + 1 = \frac{7}{3}$ or $H_1 = \frac{3}{7}$

$\frac{1}{H_2} = \frac{4}{3} + 2 = \frac{10}{3}$ or $H_2 = \frac{3}{10}$

$$\frac{1}{H_3} = \frac{4}{3} + 3 = \frac{13}{3} \qquad \text{or} \qquad H_3 = \frac{3}{13}$$

$$\frac{1}{H_4} = \frac{4}{3} + 4 = \frac{16}{3} \qquad \text{or} \qquad H_4 = \frac{3}{16}.$$

Example # 23 : Find the largest positive term of the H.P., whose first two term are $\frac{2}{5}$ and $\frac{12}{23}$.

Solution : The corresponding A.P. is $\frac{5}{2}$, $\frac{23}{12}$ ....or $\frac{30}{12}$, $\frac{23}{12}$, $\frac{16}{12}$, $\frac{9}{12}$, $\frac{2}{12}$, $\frac{-5}{12}$ ....

The H.P. is $\frac{12}{30}$, $\frac{12}{23}$, $\frac{12}{16}$, $\frac{12}{9}$, $\frac{12}{2}$, $-\frac{12}{5}$, ...

Largest positive term = $\frac{12}{2}$ = 6

Self practice problems :

(13) If a, b, c, d, e are five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. prove that a, c, e are in G.P.

(14) If the ratio of H.M. between two positive numbers 'a' and 'b' (a > b) is to their G.M. as 12 to 13, prove that a : b is 9 : 4.

(15) a, b, c are in H.P. then prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

(16) If a, b, c, d are in H.P., then show that ab + bc + cd = 3ad

Arithmetico-geometric series :

A series, each term of which is formed by multiplying the corresponding terms of an A.P. & G.P. is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3.... are in G.P..

Sum of n terms of an arithmetico-geometric series:

Let $S_n = a + (a + d)\, r + (a + 2d)\, r^2 + ... + [a + (n-1)d]\, r^{n-1}$, then

$$S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, \ r \neq 1.$$

Sum to infinity: If $|r| < 1$ & $n \to \infty$, then $\lim\limits_{n\to\infty} r^n = 0$ and $\lim\limits_{n\to\infty} n.r^n = 0$

$$\therefore \qquad S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

Example # 24 : The sum to n terms of the series $1 + 5\left(\frac{4n+1}{4n-3}\right) + 9\left(\frac{4n+1}{4n-3}\right)^2 + 13\left(\frac{4n+1}{4n-3}\right)^3 + ...$ is .

Solution : Let $x = \frac{4n+1}{4n-3}$, then

$1 - x = \frac{-4}{4n-3}$, $\frac{1}{1-x} = -\frac{(4n-3)}{4}$

$\frac{x}{1-x} = -\frac{(4n+1)}{4}$

$S = 1 + 5x + 9x^2 + ... + (4n-3)x^{n-1}$

$Sx = x + 5x^2 + ... (4n-3)x^n$

$S - Sx = 1 + 4x + 4x^2 + \ .... + 4x^{n-1} - (4n-3)x^n.$

$S(1-x) = 1 + \frac{4x}{1-x}\ [1 - x^{n-1}] - (4n-3)x^n$

$$S = \frac{1}{1-x}\left[1 + \frac{4x}{1-x} - \frac{4x^n}{1-x} - (4n-3)x^n\right] = -\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n] = n\,(4n-3).$$

Example # 25 : Find sum to infinite terms of the series $1 + 2x + 3x^2 + 4x^3 + ...,\ -1 < x < 1$

Solution : let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ....(ii)

(i) - (ii) $\Rightarrow$ $(1 - x)\, S = 1 + x + x^2 + x^3 + ...$

or $S = \dfrac{1}{(1-x)^2}$

Example # 26 : Evaluate : $1^2 + 2^2x + 3^2x^2 + 4^2x^3$ .... upto infinite terms for | x | < 1.

Solution : Let $s = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 .... \infty$ ...(i)

$xs = 1^2x + 2^2x^2 + 3^2x^3 .... \infty$ ...(ii)

(i) – (ii)

$(1 - x)\, s = 1 + 3x + 5x^2 + 7x^3 + ...$

$(1 - x)\, s = \dfrac{1}{1-x} + \dfrac{2x}{(1-x)^2}$

$s = \dfrac{1}{(1-x)^2} + \dfrac{2x}{(1-x)^3}$

$s = \dfrac{1-x+2x}{(1-x)^3}$

$s = \dfrac{1+x}{(1-x)^3}$

Self practice problems :

(17) If $4 + \dfrac{4+d}{5} + \dfrac{4+2d}{5^2}$ .... = 1, then find d.

(18) Evaluate : $1 + 3x + 6x^2 + 10x^3 + ...$ upto infinite term, where | x | < 1.

(19) Sum to n terms of the series : $1 + 2\left(1+\dfrac{1}{n}\right) + 3\left(1+\dfrac{1}{n}\right)^2 + ...$

Ans. (17) $-\dfrac{64}{5}$

(18) $\dfrac{1}{(1-x)^3}$

(19) n^2

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.) and $A \geq G \geq H$.

Example # 27 : The A.M. of two numbers exceeds the G.M. by 2 and the G.M. exceeds the H.M. by $\dfrac{8}{5}$; find the numbers.

Solution : Let the numbers be a and b, now using the relation

$G^2 = AH = (G + 2)\left(G - \dfrac{8}{5}\right) \Rightarrow G = 8 \quad ; A = 10$

i.e. $ab = 64$

also $a + b = 20$

Hence the two numbers are 4 and 16.

A.M. ≥ G.M. ≥ H.M.

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\dfrac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. = $(a_1 a_2 a_3 ....a_n)^{1/n}$ and their

H.M. = $\dfrac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}}$.

It can be shown that A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff $a_1 = a_2 = a_3 = ... = a_n$

Example # 28 : If a, b, c > 0, prove that $\dfrac{ab}{c^2} + \dfrac{bc}{a^2} + \dfrac{ca}{b^2} \geq 3$

Solution : Using the relation A.M. ≥ G.M. we have

$$\dfrac{\dfrac{ab}{c^2} + \dfrac{bc}{a^2} + \dfrac{ca}{b^2}}{3} \geq \left(\dfrac{ab}{c^2}.\dfrac{bc}{a^2}.\dfrac{ca}{b^2}\right)^{\frac{1}{3}} \Rightarrow \dfrac{ab}{c^2} + \dfrac{bc}{a^2} + \dfrac{ca}{b^2} \geq 3$$

Example # 29 : If $a_i > 0 \ \forall$ i = 1, 2, 3, ... prove that $(a_1 + a_2 + a_3 .... + a_n)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} ..... + \dfrac{1}{a_n}\right) \geq n^2$

Solution : Using the relation A.M. ≥ H.M.

$$\dfrac{a_1 + a_2 + a_3 .... + a_n}{n} \geq \dfrac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} ..... + \dfrac{1}{a_n}}$$

$$\Rightarrow \quad (a_1 + a_2 + a_3 .... + a_n)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} ..... + \dfrac{1}{a_n}\right) \geq n^2$$

Example # 30 : If x, y, z are positive then prove that $(x + y)(y + z)(z + x)\left(\dfrac{1}{x} + \dfrac{1}{y}\right)\left(\dfrac{1}{y} + \dfrac{1}{z}\right)\left(\dfrac{1}{z} + \dfrac{1}{x}\right) \geq 64$

Solution : Using the relation A.M. ≥ H.M.

$$\dfrac{x + y}{2} \geq \dfrac{2}{\dfrac{1}{x} + \dfrac{1}{y}} \Rightarrow \quad (x + y)\left(\dfrac{1}{x} + \dfrac{1}{y}\right) \geq 4 \quad .....(i)$$

similarly $(y + z)\left(\dfrac{1}{y} + \dfrac{1}{z}\right) \geq 4$ .....(ii)

$(z + x) \geq 4\left(\dfrac{1}{z} + \dfrac{1}{x}\right)$ .....(iii)

by (i), (ii) & (iii) $(x + y)(y + z)(z + x)\left(\dfrac{1}{x} + \dfrac{1}{y}\right)\left(\dfrac{1}{y} + \dfrac{1}{z}\right)\left(\dfrac{1}{z} + \dfrac{1}{x}\right) \geq 64$

Example # 31 : If n > 0, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution : Using the relation A.M. ≥ G.M. on the numbers $1, 2, 2^2, 2^3, ..., 2^{n-1}$, we have

$$\dfrac{1 + 2 + 2^2 + ... + 2^{n-1}}{n} > (1.2.\ 2^2.\ 2^3.\ ....\ .2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \quad \dfrac{2^n - 1}{2 - 1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \quad \Rightarrow \quad 2^n - 1 > n\ 2^{\frac{(n-1)}{2}}$$

$$\Rightarrow \quad 2^n > 1 + n\,2^{\frac{(n-1)}{2}}.$$

Example # 32 : If x, y, z are positive and x + y + z = 7 then find greatest value of $x^2 y^3 z^2$.

Solution : Using the relation A.M. ≥ G.M.

$$\frac{\frac{x}{2}+\frac{x}{2}+\frac{y}{3}+\frac{y}{3}+\frac{y}{3}+\frac{z}{2}+\frac{z}{2}}{7} \geq \left(\frac{x^2}{4}.\frac{y^3}{27}.\frac{z^2}{4}\right)^{\frac{1}{7}}$$

$$\Rightarrow \quad 1 \geq \left(\frac{x^2}{4}.\frac{y^3}{27}.\frac{z^2}{4}\right)^{\frac{1}{7}} \quad \Rightarrow \quad 432 \geq x^2 y^3 z^2$$

Self practice problems :

(20) If a, b, c are real and distinct, then show that $a^2 (1 + b^2) + b^2 (1 + c^2) + c^2 (1 + a^2) > 6abc$

(21) Prove that 2.4.6.8.......2n < $(n + 1)^n$. ($n \in N$)

(22) If a, b, c, d are positive real numbers prove that $\frac{bcd}{a^2}+\frac{cda}{b^2}+\frac{dab}{c^2}+\frac{abc}{d^2} > a + b + c + d$

(23) If $x^6 - 12x^5 + ax^4 + bx^3 + cx^2 + dx + 64 = 0$ has positive roots then find a, b, c, d,

(24) If a, b > 0, prove that $[(1 + a)(1 + b)]^3 > 3^3 a^2 b^2$

Ans. (23) a = 60, b = –160, c = 240, d = –192

Results :

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\, a_r = \sum_{r=1}^{n} k\, a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k + \ldots\ldots\ldots\ldots n$ times = nk; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \ldots\ldots\ldots + n^3 = \frac{n^2 (n+1)^2}{4}$

Example # 33 : Find the sum of the series to n terms whose n^{th} term is 3n + 2.

Solution : $S_n = \Sigma T_n = \Sigma(3n + 2) = 3\Sigma n + \Sigma 2 \quad = \frac{3(n+1)\, n}{2} + 2n = \frac{n}{2}(3n + 7)$

Example # 34 : $T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}(3^n - 1)$

Method of difference for finding n^{th} term :

Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1, u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + ... + u_n$ ....(i)

$S = \quad u_1 + u_2 + ... + u_{n-1} + u_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + ... + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n. So sum of series $S = \sum_{r=1}^{n} u_r$

Note : The above method can be generalised as follows :

Let u_1, u_2, u_3, ... be a given sequence.

The first differences are $\Delta_1u_1, \Delta_1u_2, \Delta_1u_3$, ... where $\Delta_1u_1 = u_2 - u_1$, $\Delta_1u_2 = u_3 - u_2$ etc.

The second differences are $\Delta_2u_1, \Delta_2u_2, \Delta_2u_3$, ..., where $\Delta_2u_1 = \Delta_1u_2 - \Delta_1u_1$, $\Delta_2u_2 = \Delta_1u_3 - \Delta_1u_2$ etc.

This process is continued untill the k^{th} differences Δ_ku_1 , Δ_ku_2 , ... are obtained, where the k^{th} differences are all equal or they form a GP with common ratio different from 1.

Case - 1 : The k^{th} differences are all equal.

In this case the n^{th} term, u_n is given by

$u_n = a_0n^k + a_1n^{k-1} + ...+ a_k$, where $a_0, a_1, ..., a_k$ are calculated by using first 'k + 1' terms of the sequence.

Case - 2 : The k^{th} differences are in GP with common ratio r $(r \neq 1)$

The n^{th} term is given by $u_n = \lambda\, r^{n-1} + a_0\, n^{k-1} + a_1\, n^{k-2} + ... + a_{k-1}$

Example # 35 : Find the n^{th} term of the series 1, 3, 8, 16, 27, 41, ...

Solution : $s = 1 + 3 + 8 + 16 + 27 + 41 + ...\ T_n$ .....(i)

$s = \quad 1 + 3 + 8 + 16 + 27\ ....T_{n-1} + T_n$ .....(ii)

(i) – (ii)

$T_n = 1 + 2 + 5 + 8 + 11 + ...\ (T_n - T_{n-1})$

$T_n = 1 + \left(\frac{n-1}{2}\right)\ [2 \times 2 + (n-2)3] = \frac{1}{2}\ [3n^2 - 5n + 4]$

Example # 36 : Find the sum to n terms of the series 5, 7, 13, 31, 85 + ...

Solution : Successive difference of terms are in G.P. with common ratio 3.

$T_n = a(3)^{n-1} + b$

$a + b = 5$

$3a + b = 7 \quad \Rightarrow \quad a = 1, b = 4$

$T_n = 3^{n-1} + 4$

$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = (1 + 3 + 3^2 + ... + 3^{n-1}) + 4n$

$\frac{1}{2}\ [3^n + 8n - 1]$

Method of difference for finding s_n :

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$t_1 = f(1) - f(0)$,

$t_2 = f(2) - f(1)$,

$\vdots \quad \vdots \quad \vdots$

$t_n = f(n) - f(n-1)$

$\Rightarrow \quad S_n = f(n) - f(0)$

Example # 37 : Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 + ...

Solution : $T_r = (3r - 1)(3r + 2) = 9r^2 + 3r - 2$

$$S_n = \sum_{r=1}^{n} T_r = 9\sum_{r=1}^{n} T_r + 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 3\left(\frac{n(n+1)}{2}\right) - 2n$$

$$= 3n(n+1)^2 - 2n$$

Example # 38 : Sum to n terms of the series $\frac{1}{(1+x)(1+3x)} + \frac{1}{(1+3x)(1+5x)} + \frac{1}{(1+5x)(1+7x)} + ...$

Solution : Let T_r be the general term of the series

$$T_r = \frac{1}{[1+(2r-1)x][1+(2r+1)x]}$$

So $$T_r = \frac{1}{2x}\left[\frac{(1+(2r+1)x)-(1+(2r-1)x)}{(1+(2r-1)x)(1+(2r+1)x)}\right] = \left[\frac{1}{(1+(2r-1)x)} - \frac{1}{(1+(2r+1)x)}\right]$$

$\therefore \quad S_n = \sum T_r = T_1 + T_2 + T_3 + ... + T_n$

$$= \frac{1}{2x}\left[\frac{1}{1+x} - \frac{1}{(1+(2n+1)x)}\right] = \frac{n}{(1+x)[1+(2n+1)x]}$$

Example # 39 : Sum to n terms of the series $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + ...$

Solution : $$T_n = \frac{1}{(3n-2)(3n+1)(3n+4)} = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\left(\frac{1}{1.4} - \frac{1}{4.7}\right) + \left(\frac{1}{4.7} - \frac{1}{7.10}\right) + ... + \frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4} - \frac{1}{(3n+1)(3n+4)}\right]$$

Example # 40 : Find the general term and sum of n terms of the series

1 + 5 + 19 + 49 + 101 + 181 + 295 + ...

Solution : The sequence of difference between successive term 4, 14, 30, 52, 80 .....

The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P>

so let nth term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ ....(i)

$8a + 4b + 2c + d = 5$ ....(ii)

$27a + 9b + 3c + d = 19$ ....(iii)

$64a + 16b + 4c + d = 49$ ....(iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \quad \Rightarrow \quad T_n = n^3 - n^2 + 1$

$$s_n = \Sigma(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

Self practice problems :

(25) Sum to n terms the following series

(i) $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + ...$

(ii) 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) ....

(iii) 4 + 14 + 30 + 52 + 82 + 114 + ...

(26) If $\sum_{r=1}^{n} T_r = (n + 1)(n + 2)(n + 3)$ then find $\sum_{r=1}^{n} \frac{1}{T_r}$

Ans. (25) (i) $\frac{2n+n^2}{(n+1)^2}$ (ii) $\frac{n(n+1)(n+2)}{6}$ (iii) $n(n + 1)^2$ (26) $\frac{n}{6(n+2)}$

Exercise-1

✍ **Marked questions are recommended for Revision.**

PART - I : SUBJECTIVE QUESTIONS

Section (A) : Arithmetic Progression

A-1. In an A.P. the third term is four times the first term, and the sixth term is 17 ; find the series.

A-2. Find the sum of first 35 terms of the series whose p^{th} term is $\frac{p}{7} + 2$.

A-3. Find the number of integers between 100 & 1000 that are divisible by 7

A-4. Find the sum of all those integers between 100 and 800 each of which on division by 16 leaves the remainder 7.

A-5.✍ The sum of first p-terms of an A.P. is q and the sum of first q terms is p, find the sum of first (p + q) terms.

A-6. The sum of three consecutive numbers in A.P. is 27, and their product is 504, find them.

A-7.✍ The fourth power of the common difference of an arithmetic progression with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.

A-8. If a, b, c are in A.P., then show that:
(i) $a^2 (b + c), b^2 (c + a), c^2 (a + b)$ are also in A.P.
(ii) $b + c – a, c + a – b, a + b – c$ are in A.P.

A-9. There are n A.M's between 3 and 54, such that the 8th mean: $(n - 2)^{th}$ mean:: 3: 5. The value of n is.

Section (B) : Geometric Progression

B-1. The third term of a G.P. is the square of the first term. If the second term is 8, find its sixth term.

B-2.✍ The continued product of three numbers in G.P. is 216, and the sum of the products of them in pairs is 156; find the numbers

B-3.✍ The sum of infinite number of terms of a G.P. is 4 and the sum of their cubes is 192. Find the series.

B-4. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 & the third is increased by 1, we obtain three consecutive terms of a G.P., find the numbers.

B-5. If the p^{th}, q^{th} & r^{th} terms of an AP are in GP. Find the common ratio of the GP.

B-6. If a, b, c, d are in G.P., prove that :
(i) $(a^2 – b^2), (b^2 – c^2), (c^2 – d^2)$ are in G.P.
(ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.

B-7. Let five geometric means are inserted between $\frac{32}{3}$ and $\frac{243}{2}$ then find sum of all the geometric means.

Section (C) : Harmonic and Arithmetic Geometric Progression

C-1. Find the 4th term of an H.P. whose 7th term is $\frac{1}{20}$ and 13th term is $\frac{1}{38}$.

C-2. Insert three harmonic means between 1 and 7.

C-3. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, r are in A.P. then prove that x, y, z are in H.P.

C-4. If a^2, b^2, c^2 are in A.P. show that b + c , c + a, a + b are in H.P.

C-5. If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$.

C-6. Sum the following series

(i) $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + ...$ to n terms.

(ii) $1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \frac{31}{256} + ...$ to infinity.

C-7. Find the sum of n terms of the series the r^{th} term of which is $(2r + 1)2^r$.

Section (D) : Relation between A.M., G.M., H.M

D-1. Using the relation A.M. ≥ G.M. prove that

(i) $(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2$. (x, y, z are positive real number)

(ii) $(a + b) . (b + c) . (c + a) > abc$; if a, b, c are positive real numbers

D-2. If x > 0, then find greatest value of the expression $\frac{x^{100}}{1 + x + x^2 + x^3 + ... + x^{200}}$.

D-3. The H.M. between two numbers is $\frac{16}{5}$, their A.M. is A and G.M. is G. If $2A + G^2 = 26$, then find the numbers.

D-4. If a, b, c are positive real numbers and sides of the triangle, then prove that
$(a + b + c)^3 \geq 27 (a + b - c)(c + a - b)(b + c - a)$

D-5. If $a_i > 0$ for all i = 1,2,3 ....n then prove that

$(1 + a_1 + a^2{}_1)(1 + a_2 + a^2{}_2) .... (1 + a_n + a^2{}_n) \geq 3^n(a_1\ a_2\ a_3 ...... a_n)$

Section (E) : Summation of series

E-1. Find the sum to n-terms of the sequence.

(i) 1 + 5 + 13 + 29 + 61 + ... up to n terms

(ii) 3 + 33 + 333 + 3333 + ... up to n terms

E-2. $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + ...$ to n terms.

E-3. (i) If $t_n = 3^n - 2^n$ then find $\sum_{n=1}^{k} t_n$.

(ii) If $t_n = n(n + 2)$ then find $\sum_{n=1}^{k} t_n$.

(iii) Find the sum to n terms of the series $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ...$

(iv) $10^2 + 13^2 + 16^2 + ...$ upto 10 terms

(v) If $\sum_{r=1}^{n} I(r) = n(2n^2 + 9n + 13)$, then find the $\sum_{r=1}^{n} \sqrt{I(r)}$

E-4. Find the sum to n-terms of the sequence.

(i) $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$

(ii) $1 . 3 . 2^2 + 2 . 4 . 3^2 + 3 . 5 . 4^2 + ...$

PART - II : ONLY ONE OPTION CORRECT TYPE

Section (A) : Arithmetic Progression

A-1. The first term of an A.P. of consecutive integer is $p^2 + 1$. The sum of $(2p + 1)$ terms of this series can be expressed as

(A) $(p + 1)^2$ (B) $(2p + 1)(p + 1)^2$ (C) $(p + 1)^3$ (D) $p^3 + (p + 1)^3$

A-2. If $a_1, a_2, a_3,........$ are in A.P. such that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$, then $a_1 + a_2 + a_3 + ... + a_{23} + a_{24}$ is equal to

(A) 909 (B) 75 (C) 750 (D) 900

A-3. If the sum of the first 2n terms of the A.P. 2, 5, 8, ..., is equal to the sum of the first n terms of the A.P. 57, 59, 61, ..., then n equals

(A) 10 (B) 12 (C) 11 (D) 13

A-4. The sum of integers from 1 to 100 that are divisible by 2 or 5 is

(A) 2550 (B) 1050 (C) 3050 (D) none of these

A-5. Let 6 Arithmetic means $A_1, A_2, A_3, A_4, A_5, A_6$ are inserted between two consecutive natural number a and b (a > b). If $A_1^2 - A_2^2 + A_3^2 - A_4^2 + A_5^2 - A_6^2$ is equal to prime number then 'b' is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Section (B) : Geometric Progression

B-1. The third term of a G.P is 4. The product of the first five terms is

(A) 4^3 (B) 4^5 (C) 4^4 (D) 4

B-2. If S is the sum to infinity of a G.P. whose first term is 'a', then the sum of the first n terms is

(A) $S\left(1-\frac{a}{S}\right)^n$ (B) $S\left[1-\left(1-\frac{a}{S}\right)^n\right]$ (C) $a\left[1-\left(1-\frac{a}{S}\right)^n\right]$ (D) $S\left[1-\left(1-\frac{S}{a}\right)^n\right]$

B-3. For a sequence $\{a_n\}$, $a_1 = 2$ and $\frac{a_{n+1}}{a_n} = \frac{1}{3}$. Then $\sum_{r=1}^{20} a_r$ is

(A) $\frac{20}{2}[4 + 19 \times 3]$ (B) $3\left(1-\frac{1}{3^{20}}\right)$ (C) $2(1 - 3^{20})$ (D) $\left(1-\frac{1}{3^{20}}\right)$

B-4. α, β be the roots of the equation $x^2 - 3x + a = 0$ and γ, δ the roots of $x^2 - 12x + b = 0$ and numbers $\alpha, \beta, \gamma, \delta$ (in this order) form an increasing G.P., then
(A) a = 3, b = 12 (B) a = 12, b = 3 (C) a = 2, b = 32 (D) a = 4, b = 16

B-5. One side of an equilateral triangle is 24 cm. The mid–points of its sides are joined to form another triangle whose mid – points are in turn joined to form still another triangle. This process continues indefinitely. Then the sum of the perimeters of all the triangles is
(A) 144 cm (B) 212 cm (C) 288 cm (D) 172 cm

B-6. Let 3 geometric means G_1, G_2, G_3 are inserted between two positive number a and b such that
$\frac{G_3 - G_2}{G_2 - G_1} = 2$, then $\frac{b}{a}$ equal to
(A) 2 (B) 4 (C) 8 (D) 16

Section (C) : Harmonic and Arithmetic Geometric Progression

C-1. If the 3rd, 6th and last term of a H.P. are , $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$ then the number of terms is equal to
(A) 100 (B) 102 (C) 99 (D) 101

C-2. If a, b, c are in H.P. then the value of $\frac{b+a}{b-a} + \frac{b+c}{b-c}$ is
(A) 1 (B) 3 (C) 4 (D) 2

C-3. If the roots of the equation $x^3 - 11x^2 + 36x - 36 = 0$ are in H.P. then the middle root is
(A) an even number (B) a perfect square of an integer
(C) a prime number (D) a composite number

C-4. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are:
(A) not in A.P./G.P./H.P. (B) in A.P.
(C) in G.P. (D) in H.P.

C-5. If $3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + ... +$ upto $\infty = 8$, then the value of d is :
(A) 9 (B) 5 (C) 1 (D) 4

C-6. Let 'n' Arithmetic Means and 'n' Harmonic Means are inserted between two positive number 'a' and 'b'. If sum of all Arithmetic Means is equal to sum of reciprocal all Harmonic means, then product of numbers is
(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) 3

C-7. Let a_1 , a_2 a_3 .... be in A.P. and h_1, h_2, h_3,...... in H.P. If $a_1 = 2 = h_1$ and $a_{30} = 25 = h_{30}$ then $(a_7 h_{24} + a_{14} h_{17})$ equal to :
(A) 50 (B) 100 (C) 200 (D) 400

C-8. **Statement 1 :** 3,6,12 are in G.P., then 9,12,18 are in H.P.
Statement 2 : If three consecutive terms of a G.P. are positive and if middle term is added in these terms, then resultant will be in H.P.
(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true

C-9. $S = 3^{10} + 3^9 + \frac{3^9}{4} + \frac{3^7}{2} + \frac{5.3^6}{16} + \frac{3^6}{16} + \frac{7.3^4}{64} + ...$upto infinite terms, then $\left(\frac{25}{36}\right)S$ equals to

(A) 6^9 (B) 3^{10} (C) 3^{11} (D) $2.\ 3^{10}$

C-10. The sum of infinite series $\frac{1.3}{2} + \frac{3.5}{2^2} + \frac{5.7}{2^3} + \frac{7.9}{2^4} + ...\infty$

(A) 21 (B) 22 (C) 23 (D) 24

Section (D) : Relation between A.M., G.M., H.M

D-1. If $x \in R$, the numbers $5^{1+x} + 5^{1-x}$, $a/2$, $25^x + 25^{-x}$ form an A.P. then 'a' must lie in the interval:

(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) [12, ∞)

D-2. If A, G & H are respectively the A.M., G.M. & H.M. of three positive numbers a, b, & c, then the equation whose roots are a, b, & c is given by :

(A) $x^3 - 3Ax^2 + 3G^3x - G^3 = 0$ (B) $x^3 - 3Ax^2 + 3(G^3/H)x - G^3 = 0$

(C) $x^3 + 3Ax^2 + 3(G^3/H)x - G^3 = 0$ (D) $x^3 - 3Ax^2 - 3(G^3/H)x + G^3 = 0$

D-3. If a, b, c, d are positive real numbers such that a + b + c + d = 2, then M = (a + b) (c + d) satisfies the relation:

(A) $0 \le M \le 1$ (B) $1 \le M \le 2$ (C) $2 \le M \le 3$ (D) $3 \le M \le 4$

D-4. If a + b + c = 3 and a > 0, b > 0, c > 0, the greatest value of $a^2b^3c^2$.

(A) $\frac{3^{10}.2^4}{7^7}$ (B) $\frac{3^9.2^4}{7^7}$ (C) $\frac{3^9.2^5}{7^7}$ (D) $\frac{3^{10}.2^5}{7^7}$

D-5. If P, Q be the A.M., G.M. respectively between any two rational numbers a and b, then P – Q is equal to

(A) $\frac{a-b}{a}$ (B) $\frac{a+b}{2}$ (C) $\frac{2ab}{a+b}$ (D) $\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right)^2$

Section (E) : Summation of series

E-1 If $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ...+ \frac{1}{n}$, then value of $1 + \frac{3}{2} + \frac{5}{3} + ... + \frac{2n-1}{n}$ is

(A) $2n - H_n$ (B) $2n + H_n$ (C) $H_n - 2n$ (D) $H_n + n$

E-2. **Statement 1 :** The sum of the first 30 terms of the sequence 1,2,4,7,11,16, 22,...... is 4520.

Statement 2 : If the successive differences of the terms of a sequence form an A.P., then general term of sequence is of the form $an^2 + bn + c$.

(A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true

E-3. The value of $\sum_{r=1}^{n} \frac{1}{\sqrt{a+r\ \ x}+\sqrt{a+(r-1)\ \ x}}$ is

(A) $\frac{n}{\sqrt{a}+\sqrt{a+nx}}$ (B) $\frac{n}{\sqrt{a}-\sqrt{a+nx}}$ (C) $\frac{\sqrt{a+nx}-\sqrt{a}}{2x}$ (D) $\frac{\sqrt{a}+\sqrt{a+n\ \ x}}{x}$

E-4. The value of ($1.1^2 + 3.2^2 + 5.3^2$ + ... + upto 10 terms) is equal to :
(A) 6050 (B) 5965 (C) 5665 (D) 5385

PART - III : MATCH THE COLUMN

1.

	Column – I	Column – II
(A)	The cofficient of x^{49} in the product $(x – 1) (x – 3) (x – 5) (x – 7)$ .... $(x – 99)$	(p) –2500
(B)	Let S_n denote sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then $\frac{S_{3n}}{S_n}$ is	(q) 9
(C)	The sum $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$ is equal to:	(r) 3/4
(D)	The length,breadth, height of a rectangular box are in G.P. (length > breadth > height) The volume is 27, the total surface area is 78. Then the length is	(s) 6

2.✍

	Column – I	Column – II
(A)	The value of xyz is 15/2 or 18/5 according as the series a, x, y, z, b are in an A.P. or H.P. then 'a + b' equals where a, b are positive integers.	(p) 2
(B)	The value of . $2^{\frac{1}{4}}\ 4^{\frac{1}{8}}\ 8^{\frac{1}{16}}. - - - \infty$ is equal to	(q) 1
(C)	If x, y, z are in A.P., then $(x + 2y – z) (2y + z – x) (z + x – y) = kxyz$, where $k \in N$, then k is equal to	(r) 3
(D)	There are m A.M. between 1 and 31. If the ratio of the 7^{th} and $(m –1)^{th}$ means is 5 : 9, then $\frac{m}{7}$ is equal to	(s) 4

Exercise-2

✍ **Marked questions are recommended for Revision.**

PART - I : ONLY ONE OPTION CORRECT TYPE

1. Given the sequence of numbers $x_1, x_2, x_3, ... x_{2013}$

which satisfy $\frac{x_1}{x_1+1} = \frac{x_2}{x_2+3} = \frac{x_3}{x_3+5} = ... = \frac{x_{2013}}{x_{2013}+4025}$, nature of the sequence is

(A) A.P. (B) G.P. (C) H.P. (D) A.G.P.

2. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. if $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{2\sqrt{3}}$ (C) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (D) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

3. If 1, 2, 3 ... are first terms; 1, 3, 5 .... are common differences and S_1, S_2, S_3 .... are sums of n terms of given p AP's; then $S_1 + S_2 + S_3 + ... + S_p$ is equal to

(A) $\frac{np(np+1)}{2}$ (B) $\frac{n(np+1)}{2}$ (C) $\frac{np(p+1)}{2}$ (D) $\frac{np(np-1)}{2}$

4. If the sum of n terms of a G.P. (with common ratio r) beginning with the p^{th} term is k times the sum of an equal number of terms of the same series beginning with the q^{th} term, then the value of k is:

(A) $r^{p/q}$ (B) $r^{q/p}$ (C) r^{p-q} (D) r^{p+q}

5. Consider the sequence 2, 3, 5, 6, 7, 8, 10, 11, ... of all positive integer, then 2011^{th} term of this sequence is

(A) 2056 (B) 2011 (C) 2013 (D) 2060

6. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where a,b,c are in AP and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x,y,z are in :

(A) HP (B) Arithmetico–Geometric Progression
(C) AP (D) GP

7. If a_1, a_2 , ... are in H.P. and $f(k) = \sum_{r=1}^{n}(a_r - a_k)$, then $\frac{a_1}{f(1)}, \frac{a_2}{f(2)}, ... \frac{a_n}{f(n)}$ are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

8. If a_1, a_2, a_3, ..., a_n are positive real numbers whose product is a fixed number c, then the minimum value of $a_1 + a_2 + a_3 + ... + a_{n-1} + 2a_n$ is

(A) $n(2c)^{1/n}$ (B) $(n + 1) c^{1/n}$ (C) $2nc^{1/n}$ (D) $(n + 1)(2c)^{1/n}$

9. The sum of the first n-terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is

(A) $\frac{n(n+1)^2}{4}$ (B) $\frac{n^2(n+2)}{4}$ (C) $\frac{n^2(n+1)}{2}$ (D) $\frac{n(n+2)^2}{4}$

10. Let T_r and S_r be the r^{th} term and sum up to r^{th} term of a series respectively. If for an odd number n, S_n = n and $T_n = \frac{T_{n-1}}{n^2}$ then T_m (m being even) is

(A) $\frac{2}{1+m^2}$ (B) $\frac{2m^2}{1+m^2}$ (C) $\frac{(m+1)^2}{2+(m+1)^2}$ (D) $\frac{2(m+1)^2}{1+(m+1)^2}$

11. If $1^2 + 2^2 + 3^2 + ... + 2003^2 = (2003)(4007)(334)$ and
(1) (2003) + (2) (2002) + (3) (2001) + ... + (2003) (1) = (2003) (334) (x)., then x equals

(A) 2005 (B) 2004 (C) 2003 (D) 2001

12. If $\sum_{r=1}^{n} t_r = \frac{n(n+1)(n+2)(n+3)}{8}$, then $\sum_{r=1}^{n} \frac{1}{t_r}$ equals

(A) $-\left(\frac{1}{(n+1)(n+2)} - \frac{1}{2}\right)$ (B) $\left(\frac{1}{(n+1)(n+2)} - \frac{1}{2}\right)$
(C) $\left(\frac{1}{(n+1)\ (n+2)} + \frac{1}{2}\right)$ (D) $\left(\frac{1}{(n-1)\ (n-2)} + \frac{1}{2}\right)$

13. If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+......$ upto $\infty = \frac{\pi^2}{6}$, then $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...... =$

(A) $\pi^2/12$ (B) $\pi^2/24$ (C) $\pi^2/8$ (D) $\pi^2/4$

14. If $S_n = \sum_{k=1}^{8n}(-1)^{\frac{k(k+1)(k+2)}{6}}(k)^2 + \sum_{k=1}^{8n}(-1)^{\frac{(k+2)(k+3)}{2}}(k^2) - 4\sum_{k=1}^{8n}(8k-2)^2$ then the value of $-S_4$ is equal to :

(A) 256 (B) 512 (C) 1024 (D) 2048

PART - II : SINGLE AND DOUBLE VALUE INTEGER TYPE

1. A man arranges to pay off a debt of Rs. 3600 by 40 annual installments which form an arithmetic series. When 30 of the installments are paid he dies leaving a third of the debt unpaid. Find the value of the first installment.

2. In a circle of radius R a square is inscribed, then a circle is inscribed in the square, a new square in the circle and so on for n times. Find the ratio of the limit of the sum of areas of all the circles to the limit of the sum of areas of all the squares as $n \to \infty$

3. If the common difference of the A.P. in which $T_7 = 9$ and $T_1T_2T_7$ is least, is 'd' then d is–

4. The number of terms in an A.P. is even ; the sum of the odd terms is 24, sum of the even terms is 30, and the last term exceeds the first by 10½; find 13th term of A.P.

5. Let AP(a, d) terms of an infinite arithmetic progression with first term 'a' and common difference 'd' ($d > 0$). If $AP(1, 3) \cap AP(3, 5) \cap AP(5, 7) = AP(m, n)$ then $|n - m|$ equals to ....

6. If $x > 0$, and $\log_2 x + \log_2(\sqrt{x}) + \log_2(\sqrt[4]{x}) + \log_2(\sqrt[8]{x}) + \log_2(\sqrt[16]{x}) + ... = 4$, then value of

$$\frac{\left(\frac{x^2+3x+2}{x+2}\right)+3x-\frac{x(x^3+1)}{(x+1)(x^2-x+1)}\log_2 8}{(x-1)(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 2)}$$ is

7. Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ the roots of the equation $Bx^2 - 6x + 1 = 0$, then find value of (A + B), such that α, β, γ & δ are in H.P.

8. Find sum of first 6 term of the infinitely decreasing G.P. whose third term, three times the product of the first and fourth term and second term form an A.P. in the indicated order, with common difference equal to 1/8.

9. If a, b, c are in GP, a – b, c – a, b – c are in HP, then the value of a + 4b + c is

10. $a, a_1, a_2, a_3, ..., a_{2n}$, b are in A.P. and $a, g_1, g_2, g_3, ...g_{2n}$, b are in G.P. and h is the harmonic mean of a and b, if $\frac{a_1+a_{2n}}{g_1g_{2n}} + \frac{a_2+a_{2n-1}}{g_2g_{2n-1}} + ... \frac{a_n+a_{n+1}}{g_ng_{n+1}} +$ is equal $\frac{Kn}{20h}$ to , then find value of K.

11. If the arithmetic mean of two numbers a & b ($0 < a < b$) is 6 and their geometric mean G and harmonic mean H satisfy the relation $G^2 + 3H = 48$. Then find the value of (2a – b)

12. If $S = \frac{5}{13} + \frac{55}{(13)^2} + \frac{555}{(13)^3} + + ...$ up to ∞, then find the value of S.

13. If $\frac{5}{k} = 1^2 - \frac{2^2}{5} + \frac{3^2}{5^2} - \frac{4^2}{5^3} + \frac{5^2}{5^4} - \frac{6^2}{5^5} + ... \infty$, then find the value of k

14. If $x_i > 0$, i = 1, 2, ..., 50 and $x_1 + x_2 + .. + x_{50} = 50$, then find the minimum value of $\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_{50}}$.

15. If a_1 , a_2 , a_3 , a_4 are positive real numbers such that $a_1 + a_2 + a_3 + a_4 = 15$ then find maximum value of $(a_1 + a_2)(a_3 + a_4)$.

16. If S_1 , S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, then is $\frac{S_3(1+8S_1)}{S_2^2}$ equal to

17. If $S = \frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + ...\infty$, then find the value of S.

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

1. The interior angles of a polygon are in A.P. If the smallest angle is 120º & the common difference is 5º, then the number of sides in the polygon is :
(A) 7 (B) 9 (C) 16 (D) 5

2. If 1, $\log_y x$, $\log_z y$, $-15 \log_x z$ are in A.P., then
(A) $z^3 = x$ (B) $x = y^{-1}$ (C) $z^{-3} = y$ (D) $x = y^{-1} = z^3$

3. If a_1, a_2 , ..., a_n are distinct terms of an A.P., then
(A) $a_1 + 2a_2 + a_3 = 0$ (B) $a_1 - 2a_2 + a_3 = 0$
(C) $a_1 + 3a_2 - 3a_3 - a_4 = 0$ (D) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

4. First three terms of the sequence 1/16, a, b, 1/6 are in geometric series and last three terms are in harmonic series if
(A) $a = \frac{1}{9}$, $b = \frac{1}{12}$ (B) $a = \frac{1}{12}$, $b = \frac{1}{9}$ (C) $a = 1$, $b = -\frac{1}{4}$ (D) $a = -\frac{1}{4}$, $b = 1$

5. Which of the following numbers is/are composite
(A)1111......1 (91 digits) (B)1111......1 (81 digits)
(C)1111......1 (75 digits) (D)1111......1 (105 digits)

6. Three numbers a, b, c between 2 and 18 are such that
(i) their sum is 25
(ii) the numbers 2, a, b, are in A.P.
(iii) the number b, c, 18 are in G.P.
then which of the following options are correct.
(A) a = 5 (B) b = 8 (C) b + c = 20 (D) a + b + c = 25

7. Consider an infinite geometric series with first term 'a' and common ratio r. If the sum is 4 and the second term is 3/4, then:
(A) $a = \frac{7}{4}$, $r = \frac{3}{7}$ (B) $a = 2$, $r = \frac{3}{8}$ (C) $a = \frac{3}{2}$, $r = \frac{1}{2}$ (D) $a = 3$, $r = \frac{1}{4}$

8. For the series $2+\left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)+\left((2\sqrt{2}-1)+\frac{1}{2}\right)+\left((3\sqrt{2}-2)+\frac{1}{2\sqrt{2}}\right)+.....$

(A) $S_n = \sqrt{2}\left(\sqrt{2}+n-1\right) -n+\left(\frac{\left(2^{n/2}-1\right)}{\left(\sqrt{2}-1\right)\ 2^{\frac{n-1}{2}}}\right)$ (B) $T_n = \sqrt{2}\left(\sqrt{2}+n-1\right) -n+\left(\frac{1}{2}\right)^{\frac{n-1}{2}}$

(C) $S_n = \frac{n}{2}\ \left(3+(n-1)\sqrt{2}-n\right)+\left(\frac{\left(2^{n/2}-1\right)}{\left(\sqrt{2}-1\right)\ 2^{\frac{n-1}{2}}}\right)$ (D) $S_n = \frac{n}{2}\ \left(3+(n-1)\sqrt{2}-n\right)+$

9.✍ If $a_k\ a_{k-1} + a_{k-1}\ a_{k-2} = 2a_k\ a_{k-2}$, $k \geq 3$ and $a_1 = 1$, here $S_p = \sum_{k=1}^{p}\frac{1}{a_k}$ and given that $\frac{S_{2p}}{S_p}$ does not depend on p then $\frac{1}{a_{2016}}$ may be

(A) 4031 (B) 1 (C) 2016 (D) 2017/2

10. If $\frac{a_{k+1}}{a_k}$ is constant for every $k \geq 1$. If $n > m \Rightarrow a_n > a_m$ and $a_1 + a_n = 66$, $a_2a_{n-1} = 128$ and $\sum_{i=1}^{n} a_i = 126$ then

(A) n = 6 (B) n = 5 (C) $\frac{a_{k+1}}{a_k} = 2$ (D) $\frac{a_{k+1}}{a_k} = 4$

11. The sides of a right triangle form a G.P. The tangent of the smallest angle is

(A) $\sqrt{\frac{\sqrt{5}+1}{2}}$ (B) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (C) $\sqrt{\frac{2}{\sqrt{5}+1}}$ (D) $\sqrt{\frac{2}{\sqrt{5}-1}}$

12. If b_1, b_2, b_3 ($b_i > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(A) r > 3 (B) 0 < r < 1 (C) r = 3.5 (D) r = 5.2

13. If a satisfies the equation $a^{2017} - 2a + 1 = 0$ and $S = 1 + a + a^2 + ... + a^{2016}$. then posible value(s) of S is/are

(A) 2016 (B) 2018 (C) 2017 (D) 2

14._ Let two term of H.P. are 12 and 15. All terms of this H.P. Consist of natural number. If among the largest possible such H.P., the three of the terms are a, b, c which are increasing G.P. then

(A) a = 15 (B) b = 30 (C) c = 60 (D) a + b + c = 2

15. Let a, x, b be in A.P; a, y, b be in G.P and a, z, b be in H.P. If x = y + 2 and a = 5z, then

(A) $y^2 = xz$ (B) x > y > z (C) a = 9, b = 1 (D) a = 1/4, b = 9/4

16._ Let x, a_1, a_2,……a_n, y, a_{n+1}, a_{n+2}, …..a_{2n}, z are in A.P.
p, g_1, g_2,…..g_n, q, g_{n+1}, g_{n+2}, …..g_{2n}, r are in G.P.
k, h_1, h_2,…..h_n, l, h_{n+1}, h_{n+2}, ….., h_{2n}, m are in H.P.

(A) If $x = p = k = 20$ and $y = q = \ell = 30$ then $z < r < m$

(B) If $y = q = \ell = 20$ and $z = r = m = 30$ then $x < p < k$

(C) If $x = p = k = 20$ and $z = r = m = 30$ then $y > q > \ell$

(D) If $x = p = k = 20$ and $z = r = m = 30$ then $y < q < \ell$

17. Which of the following is/are TRUE

(A) Equal numbers are always in A.P. , G.P. and H.P.

(B) If a, b, c be in H.P., then $a-\frac{b}{2}, \frac{b}{2}, c-\frac{b}{2}$ will be in AP

(C) If G_1 and G_2 are two geometric means and A is the arithmetic mean inserted between two positive numbers, then the value of $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$ is 2A.

(D) Let general term of a G.P. (with positive terms) with common ratio r be T_{k+1} and general term of another G.P. (with positive terms) with common ratio r be T'_{k+1}, then the series whose general term $T''_{k+1} = T_{k+1} + T'_{k+1}$ is also a G.P. with common ratio r.

18. If the arithmetic mean of two positive numbers a & b (a > b) is twice their geometric mean, then a: b is:

(A) $2+\sqrt{3} : 2-\sqrt{3}$ (B) $7+4\sqrt{3} : 1$ (C) $1 : 7-4\sqrt{3}$ (D) $2 : \sqrt{3}$

19. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, then

(A) a + c = b + d (B) e = 0

(C) a, b – 2/3, c – 1 are in A.P. (D) c/a is an integer

20. The roots of the equation $x^4 - 8x^3 + ax^2 - bx + 16 = 0$, are positive, if

(A) a = 24 (B) a = 12 (C) b = 8 (D) b = 32

21. Let $a_1, a_2, a_3, \ldots\ldots a_n$ is the sequence whose sum of first 'n' terms is represented by $S_n = an^3 + bn^2 + cn$, $n \in N$. If $a = \frac{a_1 + a_3 - xa_2}{y}$ then

(A) H.C.F of (x,y) is 2 (B) H.C.F. of (x,y) is 3

(C) L.C.M of (x,y) is 6 (D) x + y = 8

PART - IV : COMPREHENSION

Comprehension # 1 (Q.1 & 2)

We know that $1 + 2 + 3 + \ldots\ldots = \frac{n(n+1)}{2} = f(n)$,

$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = g(n)$,

$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = h(n)$

1. Even natural number which divides g(n) – f(n), for every $n \geq 2$, is

(A) 2 (B) 4 (C) 6 (D) none of these

2. f(n) + 3 g(n) + h(n) is divisible by 1 + 2 + 3 + ... + n

(A) only if n = 1 (B) only if n is odd (C) only if n is even (D) for all $n \in N$

Comprehension # 2 (Q.3 & 4)

In a sequence of (4n + 1) terms the first (2n + 1) terms are in AP whose common difference is 2, and the last (2n + 1) terms are in GP whose common ratio 0.5. If the middle terms of the AP and GP are equal, then

3. Middle term of the sequence is

(A) $\frac{n \cdot 2^{n+1}}{2^n - 1}$ (B) $\frac{n \cdot 2^{n+1}}{2^{2n} - 1}$ (C) $n \cdot 2^n$ (D) None of these

4. First term of the sequence is

(A) $\frac{4n+2n \ . \ 2^n}{2^n-1}$ (B) $\frac{4n-2n \ . \ 2^n}{2^n-1}$ (C) $\frac{2n-n \ . \ 2^n}{2^n-1}$ (D) $\frac{2n+n \ . \ 2^n}{2^n-1}$

Comprehension # 3 (Q.5 to 7)

Let $\Delta^1 T_n = T_{n+1} - T_n$, $\Delta^2 T_n = \Delta^1 T_{n+1} - \Delta^1 T_n$, $\Delta^3 T_n = \Delta^2 T_{n+1} - \Delta^2 T_n$, ... , and so on, where $T_1, T_2, T_3, \ldots T_{n-1}, T_n, T_{n+1}, \ldots$ are the terms of infinite G.P. whose first term is a natural number and common ratio is equal to 'r'.

5. If $\Delta^2 T_1 = 36$, then sum of all possible integral values of r is equal to :

(A) 8 (B) 4 (C) 5 (D) –2

6. Let $\sum_{n=1}^{\infty} T_n = \frac{7}{3}$ and $r = \frac{p}{7}$ then sum of squares of all possible value of p is equal to :

(A) 42 (B) 46 (C) 45 (D) 30

7. If $\Delta^7 T_n = \Delta^3 T_n$, then 'r' equal to

(A) 2 (B) 4 (C) 7 (D) –2

Exercise-3

Marked questions are recommended for Revision.

*** Marked Questions may have more than one correct option.**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is

[IIT-JEE - 2009, Paper-2, (3, –1), 80]

(A) $\frac{n(4n^2-1)\ c^2}{6}$ (B) $\frac{n(4n^2+1)\ c^2}{3}$ (C) $\frac{n(4n^2-1)\ c^2}{3}$ (D) $\frac{n(4n^2+1)\ c^2}{6}$

2. Let S_k, k = 1, 2,...., 100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k\ !}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100\ !} + \sum_{k=1}^{100}\left|(k^2-3k+1)\ \ S_k\right|$ is

[IIT-JEE - 2010, Paper-1, (3, 0), 84]

3. Let $a_1, a_2, a_3, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for k = 3, 4, ..., 11. If $\frac{a_1^2+a_2^2+....+a_{11}^2}{11} = 90$, then the value of $\frac{a_1+a_2+...+a_{11}}{11}$ is equal to

[IIT-JEE - 2010, Paper-2, (3, 0), 79]

4. Let $a_1, a_2, a_3, \ldots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i$, $1 \le p \le 100$.

For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

5. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} where a > 0 is

[IIT-JEE 2011, Paper-1, (4, 0), 80]

6. Let $a_1, a_2, a_3,....$ be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**
(A) 22 (B) 23 (C) 24 (D) 25

7.* Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**
(A) 1056 (B) 1088 (C) 1120 (D) 1332

8. A pack contains n card numbered from 1 to n. Two consecutive numbered card are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

9. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2 + a - 14}{a+1}$ is
[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

10. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is
[JEE (Advanced) 2015, P-2 (4, 0) / 80]

11. The least value of $\alpha \in R$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is
[JEE (Advanced) 2016, Paper-1, (3, –1)/62]
(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

12. Let $b_i > 1$ for i = 1,2,….,101. Suppose $\log_e b_1, \log_e b_2, \ldots, \log_e b_{101}$ are in Arithmetic progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, \ldots, a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \ldots + b_{51}$ and $s = a_1 + a_2 + \ldots + a_{51}$, then **[JEE (Advanced) 2016, Paper-2, (3, –1)/62]**
(A) s > t and $a_{101} > b_{101}$ (B) s > t and $a_{101} < b_{101}$
(C) s < t and $a_{101} > b_{101}$ (D) s < t and $a_{101} < b_{101}$

13. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? **[JEE(Advanced) 2017, Paper-1,(3, 0)/61]**

14. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, …., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ….. Then, the number of elements in the set $X \cup Y$ is _____. **[JEE(Advanced) 2018, Paper-1,(3, 0)/60]**

15. Let AP (a, d) denote the set of all the term of an infinite arithmetic progression with first term a and common difference d > 0. If $AP(1, 3) \cap AP(2, 5) \cap AP(3, 7) = AP(a, d)$ then a + d equals………
[JEE(Advanced) 2019, Paper-1,(4, –1)/62]

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = ... = a_{10} = 150$ and $a_{10}, a_{11},....$ are in an AP with common difference –2, then the time taken by him to count all notes is **[AIEEE 2010 (8, –2), 144]**
(1) 34 minutes (2) 125 minutes (3) 135 minutes (4) 24 minutes

2. A man saves Rs. 200 in each of the first three months of his service. In each of the subsequent months his saving increases by Rs. 40 more than the saving of immediately previous month. His total saving from the start of service will be Rs. 11040 after : **[AIEEE 2011, I, (4, –1), 120]**
(1) 18 months (2) 19 months (3) 20 months (4) 21 months

3. Let a_n be the n^{th} term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is : **[AIEEE 2011, II, (4, –1), 120]**
(1) $\alpha - \beta$ ((2) $\frac{\alpha-\beta}{100}$ (3) $\beta - \alpha$ (4) $\frac{\alpha-\beta}{200}$

4. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777,....., is **[AIEEE - 2013, (4, –1),360]**
(1) $\frac{7}{81}(179 - 10^{-20})$ (2) $\frac{7}{9}(99 - 10^{-20})$ (3) $\frac{7}{81}(179 + 10^{-20})$ (4) $\frac{7}{9}(99 + 10^{-20})$

5. If $(10)^9 + 2(11)^1 (10)^8 + 3(11)^2 (10)^7 + \ldots\ldots\ldots + 10 (11)^9 = k(10)^9$, then k is equal to **[JEE(Main) 2014, (4, – 1), 120]**
(1) 100 (2) 110 (3) $\frac{121}{10}$ (4) $\frac{441}{100}$

6. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is **[JEE(Main) 2014, (4, – 1), 120]**
(1) $2 - \sqrt{3}$ (2) $2 + \sqrt{3}$ (3) $\sqrt{2} + \sqrt{3}$ (4) $3 + \sqrt{2}$

7. If m is the A. M. of two distinct real numbers l and n(l, n > 1) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals : **[JEE(Main) 2015, (4, – 1), 120]**
(1) $4 l^2 mm$ (2) $4 lm^2 n$ (3) $4 lmn^2$ (4) $4 l^2m^2n^2$

8. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots\ldots$ is : **[JEE(Main) 2015, (4, – 1), 120]**
(1) 71 (2) 96 (3) 142 (4) 192

9. If the 2^{nd}, 5^{th} and 9^{th} terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is: **[JEE(Main) 2016, (4, – 1), 120]**
(1) $\frac{4}{3}$ (2) 1 (3) $\frac{7}{4}$ (4) $\frac{8}{5}$

10. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \ldots\ldots$ is $\frac{16}{5}$ m, then m is equal to : **[JEE(Main) 2016, (4, – 1), 120]**
(1) 101 (2) 100 (3) 99 (4) 102

11. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, Then **[JEE(Main) 2017, (4, – 1), 120]**
(1) b , c and a are in G.P. (2) b, c and a are in A.P.
(3) a, b and c are in A.P. (4) a, b and c are in G.P.

12. Let a,b,c $\in$ R. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $\forall$ x,y $\in$ R, then $\sum_{n=1}^{10} f(n)$ is equal to **[JEE(Main) 2017, (4, – 1), 120]**
(1) 330 (2) 165 (3) 190 (4) 225

13. If, for a positive integer n, the quadratic equation, $x(x + 1) + (x + 1)(x + 2) + \ldots\ldots + (x + \overline{n-1})(x + n) = 10n$ has two consecutive integral solutions, then n is equal to **[JEE(Main) 2017, (4, – 1), 120]**
(1) 12 (2) 9 (3) 10 (4) 11

14. Let $a_1, a_2, a_3, \ldots\ldots, a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \ldots\ldots + a_{17}^2 = 140\ m$, then m is equal to : **[JEE(Main) 2018, (4, – 1), 120]**

(1) 34 (2) 33 (3) 66 (4) 68

15. Let A be the sum of the first 20 terms and B be sum of the first 40 terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.\ 6^2 + \ldots..$ If B – 2A = 100 λ, then λ is equal to :

[JEE(Main) 2018, (4, – 1), 120]

(1) 464 (2) 496 (3) 232 (4) 248

16. The sum of the following series

$1 + 6 + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \frac{15(1^2 + 2^2 + \ldots.. + 5^2)}{11} + \ldots.$ up to 15 terms, is :

[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]

(1) 7510 (2) 7830 (3) 7520 (4) 7820

17. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to :

[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]

(1) $\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $\frac{7}{4}$ (4) $\frac{5}{4}$

18. The sum of all natural number 'n' such that 100 < n < 200 and H.C.F. (91,n) > 1 is :

[JEE(Main) 2019, Online (08-04-19),P-1 (4, – 1), 120]

(1) 3303 (2) 3203 (3) 3221 (4) 3121

19. Let $a_1, a_2, a_3, \ldots..$ be an A.P. with $a_6 = 2$. Then the common difference of this A.P., which maximizes the product $a_1\ a_4\ a_5$ is : **[JEE(Main) 2019, Online (10-04-19),P-2 (4, – 1), 120]**

(1) $\frac{3}{2}$ (2) $\frac{8}{5}$ (3) $\frac{6}{5}$ (4) $\frac{2}{3}$

20. Let a,b and c be in G.P. with common ratio r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. Ifs 3a, 7b and 15c are the first three terms of an A.P., then the 4th therm this A.P. is :

[JEE(Main) 2019, Online (10-04-19),P-2 (4, – 1), 120]

(1) $\frac{7}{3}a$ (2) a (3) $\frac{2}{3}a$ (4) 5a

21. The greatest positive integer k, for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + \ldots. + 49^2 + 49 + 1$, is **[JEE(Main) 2020, Online (07-01-20),P-1 (4, – 1), 120]**

(1) 32 (2) 63 (3) 65 (4) 60

22. Let f : R $\rightarrow$ R be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, f(x) and $(3^x + 3^{-x})$ are in A.P., then the minimum value of f(x) is : **[JEE(Main) 2020, Online (08-01-20),P-1 (4, –1), 120]**

(1) 0 (2) (3) 4 (4) 3

23. Let a_n be the nth term of a G.P. of positive terms. If $\sum_{n=1}^{100} a_{2n+1} = 200$ and $\sum_{n=1}^{100} a_{2n} = 100$, then $\sum_{n=1}^{200} a_n$ is equal to : **[JEE(Main) 2020, Online (09-01-20),P-2 (4, –1), 120]**

(1) 175 (2) 150 (3) 300 (4) 225

24. If x = $\sum_{n=0}^{\infty} (-1)^n \tan^{2n}\theta$ and y= $\sum_{n=0}^{\infty} \cos^{2n}\theta$, for $0 < \theta < \frac{\pi}{4}$, then:

(1) y(1–x) = 1 (2) y(1+x) = 1 (3) x(1+y) = 1 (4) x(1–y) = 1

Answers

EXERCISE - 1

PART - I

Section (A) :

A-1. 2, 5, 8,..... **A-2.** 160 **A-3.** 128 **A-4.** 19668 **A-5.** –(p + q) **A-6.** 4, 9, 14
A-9. 16

Section (B) :

B-1. 128 **B-2.** 2, 6, 18 or 18, 6, 2 **B-3.** 6, –3, 3/2, ... **B-4.** 3, 7, 11 or 12, 7, 2
B-5. $\frac{q-r}{p-q}$ **B-7.** 211

Section (C) :

C-1. $\frac{1}{11}$ **C-2.** $\frac{14}{11}, \frac{14}{8}, \frac{14}{5}$ **C-6.** (i) $4 - \frac{2+n}{2^{n-1}}$ (ii) $\frac{8}{3}$
C-7. $n.2^{n+2} - 2^{n+1} + 2$.

Section (D) :

D-2. $\frac{1}{201}$ **D-3.** 2, 8

Section (E) :

E-1. (i) $2^{n+2} - 3n - 4$ (ii) $\frac{1}{27}$ $(10^{n+1} - 9n - 10)$ **E-2.** $\frac{n.2^n - 2^n + 1}{2^n}$

E-3. (i) $\frac{1}{2}$ $(3^{k+1} + 1) - 2^{k+1}$ (ii) $\frac{1}{6}k(k+1)(2k+7)$

(iii) $-\frac{n(n+1)}{2}$ if n is even, $\frac{n(n+1)}{2}$ if n is odd (iv) 6265 (v) $\sqrt{\frac{3}{2}}(n^2+3n)$

E-4. (i) $\frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$ (ii) $\frac{n}{10}$ $(n+1)(n+2)(n+3)(2n+3)$

PART - II

Section (A) :
A-1. (D) **A-2.** (D) **A-3.** (C) **A-4.** (C) **A-5.** (C)

Section (B) :
B-1. (B) **B-2.** (B) **B-3.** (B) **B-4.** (C) **B-5.** (A) **B-6.** (D)

Section (C) :
C-1. (A) **C-2.** (D) **C-3.** (C) **C-4.** (D) **C-5.** (A) **C-6.** (A) **C-7.** (B)
C-8. (A) **C-9.** (B) **C-10.** (C)

Section (D) :

D-1. (D) **D-2.** (B) **D-3.** (A) **D-4.** (A) **D-5.** (D)

Section (E) :
E-1 (A) **E-2.** (D) **E-3.** (A) **E-4.** (C)

PART - III

1. (A) → (p), (B) → (s), (C) → (r), (D) → (q)
2. (A) → (s), (B) → (p), (C) → (s), (D) → (p)

EXERCISE - 2

PART - I

1.	(A)	**2.**	(D)	**3.**	(A)	**4.**	(C)	**5.**	(A)
6.	(A)	**7.**	(C)	**8.**	(A)	**9.**	(C)	**10.**	(D)
11.	(A)	**12.**	(A)	**13.**	(C)	**14.**	(C)		

PART - II

1.	51.00	**2.**	00.78	**3.**	01.65	**4.**	19.50	**5.**	02.00
6.	01.66 or 01.67	**7.**	11.00	**8.**	01.96 or 01.97	**9.**	00.00	**10.**	40.00
11.	00.00	**12.**	01.80	**13.**	10.80	**14.**	50.00	**15.**	56.25
16.	09.00	**17.**	00.50						

PART - III

1.	(B)	**2.**	(ABCD)	**3.**	(BD)	**4.**	(BD)	**5.**	(ABCD)
6.	(ABCD)	**7.**	(D)	**8.**	(BC)	**9.**	(AB)	**10.**	(AC)
11.	(BC)	**12.**	(ABCD)	**13.**	(CD)	**14.**	(ABC)	**15.**	(ABC)
16.	(ABC)	**17.**	(CD)	**18.**	(ABC)	**19.**	(ABCD)	**20.**	(AD)
21.	(ACD)								

PART - IV

1.	(A)	**2.**	(D)	**3.**	(A)	**4.**	(B)	**5.**	(A)
6.	(B)	**7.**	(A)						

EXERCISE - 3

PART - I

1.	(C)	**2.**	3	**3.**	0

4. 3 or 9, both 3 and 9 (The common difference of the arithmatic progression can be either 0 or 6, and accordingly the second term can be either 3, or 9 ; thus the answers 3, or 9, or both 3 and 9 are acceptable.)

5.	(8)	**6.**	(D)	**7.**	(A,D)	**8.**	5	**9.**	(4)
10.	9	**11.**	(C)	**12.**	(B)	**13.**	(6)	**14.**	(3748)
15.	(157)								

PART - II

1.	(1)	**2.**	(4)	**3.**	(2)	**4.**	(3)	**5.**	(1)
6.	(2)	**7.**	(2)	**8.**	(2)	**9.**	(1)	**10.**	(1)
11.	(2)	**12.**	(1)	**13.**	(4)	**14.**	(1)	**15.**	(4)
16.	(4)	**17.**	(3)	**18.**	(4)	**19.**	(2)	**20.**	(2)
21.	(2)	**22.**	(4)	**23.**	(2)	**24.**	(1)		

High Level Problems (HLP)

SUBJECTIVE QUESTIONS

1. Prove that $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be terms of a single A.P.

2. If the sum of the first m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that $(m + n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m + p)\left(\frac{1}{m}-\frac{1}{n}\right)$.

3. If a and b are p^{th} and q^{th} terms of an AP, then find the sum of its (p + q) terms

4. In an A.P. of which 'a' is the Ist term, if the sum of the Ist ' p ' terms is equal to zero, show that the sum of the next ' q ' terms is $-\frac{a\,(p+q)\,q}{p-1}$.

5. If $\frac{a+be^y}{a-be^y} = \frac{b+ce^y}{b-ce^y} = \frac{c+de^y}{c-de^y}$, then show that a,b,c,d are in G.P.

6. The sum of the first ten terms of an AP is 155 & the sum of first two terms of a GP is 9. The first term of the AP is equal to the common ratio of the GP & the first term of the GP is equal to the common difference of the AP. Find the two progressions.

7. Find the sum in the n^{th} group of sequence,

(i) (1), (2, 3); (4, 5, 6, 7); (8, 9,........, 15); ....

(ii) (1), (2, 3, 4), (5, 6, 7, 8, 9),........

8. Let a, b be positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, show that

$$\frac{G_1\,G_2}{H_1\,H_2} = \frac{A_1+A_2}{H_1+H_2} = \frac{(2a+b)\,(a+2b)}{9\,ab}.$$

9. If total number of runs scored in n matches is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$ where n > 1 and the runs scored in the k^{th} match are given by k. 2^{n+1-k}, where $1 \le k \le n$, find n

10. Let a_1, a_2,........, a_n be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively the arithmetic mean, geometric mean & harmonic mean of a_1, a_2,......, a_n. Prove that $G = \prod_{K=1}^{n}(A_k \quad H_k)^{\frac{1}{2n}}$, Where G is geometric mean between G_1, G_2, ..., G_n.

11. If a, b, c are in A.P., p, q, r are in H.P. and ap, bq, cr are in G.P., then find $\frac{p}{r} + \frac{r}{p}$.

12. If the sum of the roots of the quadratic equation, $ax^2 + bx + c = 0$ is equal to sum of the squares of their reciprocals, then prove that $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

13. If a, b, c are in H.P.; b, c, d are in G.P.; and c, d, e are in A.P. such that $(ka - b)^2 e = ab^2$ then value of k.

14. The value of x + y + z is 15 if a, x, y, z, b are in AP while the value of (1/x) + (1/y) + (1/z) is 5/3 if a, x, y, z, b are in HP. Find a and b.

15. If n is a root of the equation $x^2 (1 - ac) - x (a^2 + c^2) - (1 + ac) = 0$ and if n HM's are inserted between a and c, show that the difference between the first and the last mean is equal to ac(a – c).

16. If a, b, c are positive real numbers, then prove that

(i) $b^2c^2 + c^2a^2 + a^2b^2 \geq abc\,(a + b + c)$.

(ii) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

(iii) $\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{9}{a+b+c}$

17. Solve the equation $(2 + x_1 + x_2 + x_3 + x_4)^5 = 6250\, x_1 x_2 x_3 x_4$ .where $x_1, x_2, x_3, x_4 > 0$.

18. Let $a_1, a_2, ...a_n$, be real numbers such that

$$\sqrt{a_1} + \sqrt{a_2 - 1} + \sqrt{a_3 - 2} + ... + \sqrt{a_n - (n-1)} = \frac{1}{2}\,(a_1 + a_2 + ...+ a_n) - \frac{n(n-3)}{4}$$

then find the value of $\sum_{i=1}^{100} a_i$

19. If $a_i \in R$, i = 1, 2, 3, ...n and all a_i's are distinct such that $\left(\sum_{i=1}^{n-1} a_i^2\right) + 6\left(\sum_{i=1}^{n-1} a_i \quad a_{i+1}\right) + 9\sum_{i=2}^{n} a_i^2 \leq 0$ and $a_1 = 8$ then find the sum of first five terms.

20. Let $\{a_n\}$ and $\{b_n\}$ are two sequences given by $a_n = (x)^{1/2^n} + (y)^{1/2^n}$ and $b_n = (x)^{1/2^n} - (y)^{1/2^n}$ for all $n \in N$. Then find $a_1a_2a_3........a_n$.

21. Given that $a_1, a_2, a_3, ...a_n$ form an A.P. find then following sum $\sum_{i=1}^{10} \frac{a_i a_{i+1} a_{i+2}}{a_i + a_{i+2}}$

Given that $a_1 = 1$; $a_2 = 2$

22. Find sum of the series $\frac{n}{1\,.\,2\,.\,3} + \frac{n-1}{2\,.\,3\,.\,4} + \frac{n-2}{3\,.\,4\,.\,5} + \ldots$ up to n terms..

23. Find the value of $S_n = \sum_{n=1}^{n} \frac{3^n\ .\ 5^n}{(5^n - 3^n)(5^{n+1} - 3^{n+1})}$ and hence S_∞.

24. Circles are inscribed in the acute angle α so that every neighbouring circles touch each other. If the radius of the first circle is R, then find the sum of the radii of the first n circles in terms of R and α.

25. Let A, G, H be A.M., G.M. and H.M. of three positive real numbers a, b, c respectively such that $G^2 = AH$, then prove that a, b, c are terms of a GP.

26. If $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6} n\ (2n^2 + 9n + 13)$, then $\sum_{r=1}^{\infty} \frac{1}{r\ .\ \sqrt{t_r}}$ equals

27. In the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, $\Delta = b^2 - 4ac$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P. where α, β are the root of $ax^2 + bx + c = 0$, then prove that $c\Delta = 0$

28. If sum of first n terms of an A.P. (having positive terms) is given by $S_n = (1 + 2T_n)(1 - T_n)$ where T_n is the n^{th} term of series, then $T_2^{\,2} = \frac{\sqrt{a} - \sqrt{b}}{4}$, $(a \in N, b \in N)$, then find the value of (a + b)

Answers

3. $\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$

6. (3 + 6 + 12 +......) ; (2/3 + 25/3 + 625/6 +......) G.P.

(2 + 5 + 8 + ...) ; $\left(\frac{25}{2}+\frac{79}{6}+\frac{83}{6}+..........\right)$ A.P.

7. (i) $2^{n-2}(2^n + 2^{n-1} - 1)$ (ii) $(n-1)^3 + n^3$

9. 7

11. $\frac{a}{c}+\frac{c}{a}$

13. 2

14. a = 1, b = 9 **OR** b = 1, a = 9

18. 5050

19. $\frac{488}{81}$

20. $\frac{x-y}{b_n}$

21. $\frac{495}{2}$

22. $\frac{n(n+1)}{4(n+2)}$

23. $\frac{3}{4}$

24. $\frac{R\left(1-\sin\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}}\left[\left(\frac{1+\sin\frac{\alpha}{2}}{1-\sin\frac{\alpha}{2}}\right)^n-1\right]$

26. 1

28. 6

SOLUTIONS OF SEQUENCE & SERIES

EXERCISE - 1

PART-I

Section (A) :

A-1. (a + 2d) = 4a $\Rightarrow$ 3a = 2d.

We have given that a + 5d = 17 $\Rightarrow$ $a + 5\left(\frac{3a}{2}\right) = 17$

a = 2, d = 3

so series 2, 5, 8 ....

A-2. Given that; $T_p = \frac{p}{7} + 2$, then $S_p = \frac{1}{7}\sum p + \sum 2 = \frac{p(p+1)}{14} + 2p$.

taking p = 35 $S_{35} = 160$

A-3. 994 = 105 + (n –1) 7 $\Rightarrow$ 889 + 7 = 7n $\Rightarrow$ n = 128

A-4. First No. = 103 last No. = 791

No. of terms = 44 $s = \frac{44}{2}$ [103 + 791] = 22 [894] = 19668

A-5. $q = \frac{p}{2}[2A + (p-1)d] \Rightarrow \frac{2q}{p} = 2A + (p-1)d$ .....(i)

$p = \frac{p}{2}[2A + (q-1)d] \Rightarrow \frac{2q}{p} = 2A + (q-1)d$ .....(ii)

on subtracting equation (i) from (ii), we get

$\frac{2}{pq}(q^2 - p^2) = (p-q)\,d \Rightarrow d = \frac{-2}{pq}(p+q)$

$\therefore$ Sum of (p + q) terms is

$\frac{p+q}{2} = [2A + (p+q-1)d] = \frac{p+q}{2}[2A + (p-1)d + qd] = \frac{p+q}{2}\left[\frac{2q}{p} + q\left\{\frac{-2}{pq}(p+q)\right\}\right]$

$= \frac{p+q}{2}\left[\frac{2q}{p} - 2 - \frac{2q}{p}\right] = -(p+q)$ **Ans.**

A-6. Numbers are a – d, a, a + d $\Rightarrow$ s = a – d + a + a + d = 27 $\Rightarrow$ a = 9

$a(a^2 - d^2) = 504 \Rightarrow 9(81 - d^2) = 504 \Rightarrow 81 - d^2 = 56$

$d^2 = 25 \Rightarrow d = \pm 5$. Numbers are 4, 9, 14

A-7. $\because$ $(a-3d)(a-d)(a+d)(a+3d) + 16d^4 = (a^2 - 9d^2)(a^2 - d^2) + 16d^4$

$= a^4 - a^2d^2 - 9a^2d^2 + 9d^4 + 16d^4 = a^4 - 10a^2d^2 + 25d^4 = [a^2 - 5d^2]^2$

$= (a^2 - d^2 - 4d^2)^2 = ((a-d)(a+d) - (2d)^2)^2$

$\because$ (a – d), (a + d), 2d are integers. Hence Proved

A-8. (i) a, b, c are in A.P. a (ab + bc + ac), b(ab + bc + ac), c(ab + bc + ac) are in A.P.

$\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

(ii) b + c – a, a + c – b, a + b – c are in A.P

$\Rightarrow$ 2(a + c –b) = (b + c – a) (a + b – c) $\Rightarrow$ a + c = 2b $\Rightarrow$ a , b , c are in A.P

A-9. $\frac{(54-3)}{n+1}=d$; $d=\frac{51}{n+1}$

$\frac{A_8}{A_{n-2}}=\frac{3}{5}$ $\Rightarrow$ $\frac{3+8\frac{51}{n+1}}{3+(n-2)\frac{51}{n+1}}=\frac{3}{5}$ $\Rightarrow$ $\frac{3n+3+408}{3n+3+51n-102}=\frac{3}{5}$

$\Rightarrow$ 15n + 2055 = 162n – 297 $\Rightarrow$ 147 n = 2352 ; n = 16

Section (B) :

B-1. Let the three terms be a, ar, ar^2 $\Rightarrow$ $ar^2 = a^2$ $\Rightarrow$ $a = r^2$ and ar = 8

$\Rightarrow$ $r^3 = 8$, r = 2 and a = 4 $\therefore$ $T_6 = 4(2)^5 = 128$

B-2. Let the Numbers are $\frac{a}{r}$, a, ar so $a^3 = 216$ $\Rightarrow$ a = 6 $\Rightarrow$ $\frac{a}{r}.a + a.ar + ar.\frac{a}{r} = 156$

$\Rightarrow$ $a^2(1+r+\frac{1}{r}) = 156$ $\Rightarrow$ $(1+r+\frac{1}{r}) =$ $\Rightarrow 1+r+\frac{1}{r}=\frac{156}{36}$

$\Rightarrow$ r = 3 or 1/3. Numbers are 2, 6, 18 or 18, 6, 2

B-3. $\frac{a}{1-r}=4$ $\Rightarrow$ $\frac{a^3}{1-r^3}=192$ $\Rightarrow$ $\frac{(1-r)^3}{1-r^3}=\frac{192}{(4)^3}$ $\Rightarrow$ $\frac{(1-r)^2}{1+r+r^2}=3$

$\Rightarrow$ $1 + r^2 - 2r = 3 + 3r + 3r^2$ $\Rightarrow$ $2r^2 + 5r + 2 = 0$ $\Rightarrow$ (2r + 1) (r + 2) = 0

r = –1/2, r = –2(rejected) When r = –1/2 , a = 6 so series is 6, –3, 3/2 .....

B-4. Let a – d , a, a + d $\Rightarrow$ 3a = 21 $\Rightarrow$ a =7

a – d, a –1, a + d + 1 are in G.P $\Rightarrow$ 7 – d, 6 , 8 + d are in G.P

$\Rightarrow$ 36 = (7– d) (8 + d) $\Rightarrow$ $36 = 56 - d - d^2$

$\Rightarrow$ $d^2 + d - 20 = 0$ $\Rightarrow$ d = –5 , 4

so Numbers are 3, 7, 11 $\Rightarrow$ 12, 7, 2

B-5. $\frac{T_q}{T_p}=\frac{T_r}{T_q}$ = common ratio; $\frac{a+(q-1)d}{a+(p-1)d}=\frac{a+(r-1)d}{a+(q-1)d}$ using dividendo

$\frac{(q-p)}{a+(p-1)d}=\frac{(r-q)}{a+(q-1)d}$ $\Rightarrow$ $\frac{T_q}{T_p}=\frac{r-q}{q-p}=\frac{q-r}{p-q}$

B-6. (i) Let b = ar

$c = ar^2$ and $d = ar^3$

So $a^2(1-r^2)$, $a^2(r^2)(1-r^2)$, $a^2r^4(1-r^2)$ these are in G.P.

So (a^2-b^2), (b^2-c^2), (c^2-d^2) are in G.P.

(ii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2} = \frac{1}{a^2(1+r^2)}, \frac{1}{a^2r^2(1+r^2)}, \frac{1}{a^2r^4(1+r^2)}$ are in G.P.

B-7. Common ratio of means = $\left(\frac{243}{2}\times\frac{3}{32}\right)^{1/6}=\frac{3}{2}$

$\Rightarrow$ means are 16, 24, 36, 54, 81

their sum is 211.

Section (C) :

C-1. $T_7=\frac{1}{20}$ $\Rightarrow$ a + 6d = 20 ; $T_{13}=\frac{1}{38}$ $\Rightarrow$ a + 12d = 38

d = 3, a = 2 so $T_4=\frac{1}{2+9}=\frac{1}{11}$

C-2. $1, A_1, A_2, A_3, \frac{1}{7}$

$\frac{1}{7} = 1 + 4 .d$

$d = \frac{\frac{1}{7}-1}{4} = \frac{-6}{28} = \frac{-3}{14}$

$A_1 = 1 - \frac{3}{14} = \frac{11}{14}$

$A_2 = 1 - \frac{6}{14} = \frac{18}{14}$

$A_3 = 1 - \frac{9}{14} = \frac{5}{14}$

so $\frac{14}{11}, \frac{14}{8}, \frac{14}{5}$ are three harmonic means

C-3. Let $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = k$

$P = \frac{a-x}{kx}, q = \frac{a-y}{kx}, r = \frac{a-z}{kz}$

$2\left(\frac{a-y}{ky}\right) = \frac{a-x}{kx} + \frac{a-z}{kz}$

$2\left(\frac{a}{y}-1\right) = \frac{a}{x} - 1 + \frac{a}{z} - 1$

$\frac{2a}{y} = \frac{a}{x} + \frac{a}{z}$

$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

Hence x, y, z are in H.P.

C-4. a^2, b^2, c^2 are in A.P.
Let b + c, c + a, a + b are in H.P.

then $\frac{1}{b+c}\ \frac{1}{c+a}\ \frac{1}{a+b}$ are in A.P.

$\frac{2}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$

$2b^2 = a^2 + c^2$

hence a^2, b^2, c^2 are in A.P.

if a^2 , b^2, c^2 are in A.P. then b + c, c + a, a – b are in H.P.

C-5. $b = \frac{2ac}{a+c}$

$\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$

L.H.S. = $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{\frac{2ac}{a+c}-a} + \frac{1}{\frac{2ac}{a+c}-c} = \frac{(a+c)}{a(2c-a-c)} + \frac{(a+c)}{c(2a-a-c)}$

$= \frac{a+c}{a(c-a)} + \frac{(a+c)}{c(a-c)} = \frac{a+c}{(c-a)}\left[\frac{1}{a} - \frac{1}{c}\right] = \frac{a+c}{ac} = \frac{1}{a} + \frac{1}{c}$ = RHS

C-6. (i) $1 + \frac{2}{2} + \frac{3}{2^2} + \ldots\ldots\ldots$ n terms

$T_n = \frac{n}{2^{n-1}}$

$S = 1 + \frac{2}{2} + \frac{3}{2^2} + \ldots\ldots\ldots + \frac{n}{2^{n-1}}$ ...(i)

$\frac{1}{2}S = \frac{1}{2} + \frac{2}{2^2} + \ldots\ldots\ldots + \frac{(n-1)}{2^{n-1}} + \frac{n}{2^n}$ ...(ii)

(i) – (ii) we get

$\frac{1}{2}S = [1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}] - \frac{n}{2^n}$

$\frac{1}{2}S = \frac{1.\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} - \frac{n}{2^n} \Rightarrow S = 4 - 4\left(\frac{1}{2}\right)^n - \frac{2n}{2^n}$; $S = 4 - \frac{2+n}{2^{n-1}}$

(ii) $S = 1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \frac{31}{256} + \ldots\ldots\infty$ ..... (i)

$\frac{1}{4}S = \frac{1}{4} + \frac{3}{16} + \frac{7}{64} + \ldots\ldots\infty$ ..... (ii)

(i) – (ii), we get

$\frac{3}{4}S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\infty \Rightarrow \frac{3}{4}S = \frac{1}{1/2} \Rightarrow S = \frac{8}{3}$

C-7. $T_r = (2r + 1)\, 2^r$

$S = 3.2 + 5.2^2 + 7.2^3 + \ldots\ldots + (2n + 1)\, 2^n$ ...(i)

$2S = 3.2^2 + 5.2^3 + \ldots\ldots + (2n + 1)\, 2^n + (2n + 1)\, 2^{n-1}$ ...(ii)

(i) – (ii) we get

$-S = 3.2 + (2.2^2 + 2.2^3 + \ldots\ldots + 2.2^n) - (2n + 1)\, 2^{n+1} \Rightarrow -S = 6 + 8\,(2^{n-1} - 1) - (2n + 1)\, 2^{n+1}$

$S = 2 - 2^{n+2} + n.\, 2^{n+2} + 2^{n+1} \Rightarrow S = n.2^{n+2} - 2^{n+1} + 2$

Section (D) :

D-1. (i) $(x^2y + y^2z + z^2x)\,(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2 \Rightarrow \frac{x^2y + y^2z + z^2x}{3} \geq (x^2y.\, y^2z.\, z^2x)^{1/3}$

$x^2y + y^2z + z^2x \geq 3xyz$ ......(i)

and $\frac{xy^2 + yz^2 + zx^2}{3} \geq (xy^2 .\, yz^2 .\, zx^2)^{1/3} \Rightarrow xy^2 + yz^2 + zx^2 \geq 3xyz$ ......(ii)

By (i) and (ii)

$\Rightarrow (x^2y + y^2z + z^2x)\,(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2$

(ii) $(a + b)\,(b + c)\,(c + a) > abc$

$\frac{abc + b^2c + bc^2 + c^2a + a^2b + ab^2 + abc + a^2c}{8} \geq (abc.\, b^2c.\, bc^2.\, c^2a.\, a^2b.\, ab^2.\, abc.\, a^2c)$

$\Rightarrow (a + b)\,(b + c)\,(c + a) \geq 8abc \Rightarrow (a + b)\,(b + c)\,(c + a) > abc$

D-2. $\frac{x^{100}}{1 + x + x^2 + x^3 + \ldots.. + x^{200}}$

AM $\geq$ GM

$\frac{1 + x + x^2 + x^3 + \ldots.. + x^{200}}{201} \geq \left(1\ .\ x\ .\ x^2\ldots..x^{200}\right)^{\frac{1}{201}} \Rightarrow \frac{1 + x + x^2 + x^3 + \ldots.. + x^{200}}{201} \geq \left(x^{\frac{201}{2}\,.\,200}\right)^{\frac{1}{201}}$

$\frac{1 + x + x^2 + x^3 + \ldots.. + x^{200}}{201} \geq x^{100} \Rightarrow \frac{x^{100}}{1 + x + x^2 + x^3 + \ldots.. + x^{200}} \leq \frac{1}{201}$

D-3. Let a and b are two numbers $\frac{2ab}{a+b} = \frac{16}{5}$ ..... (1)

$\frac{a+b}{2} = A$ and $\sqrt{ab} = G$

$\because$ $2A + G^2 = 26$ $\Rightarrow$ $(a + b) + ab = 26$ ..... (2)

$\Rightarrow$ $\frac{10\ \ ab}{16} + ab = 26$ $\Rightarrow$ $26\ ab = 26 \times 16 \Rightarrow$ $ab = 16$

$\therefore$ from (1), we get $a + b = 10$ So a, b are (2, 8) **Ans.**

D-4. $\frac{(a+b-c)+(a+c-b)+(b+c-a)}{3} \geq ((a + b - c)\ (c + a - b)\ (b + c - a))^{1/3}$

$(a+b+c) \geq 3((a + b - c)\ (c + a - b)\ (b + c - a))^{1/3}$

$(a+b+c)^3 \geq 27\ (a + b - c)\ (c + a - b)\ (b + c - a)$ Hence Proved

D-5. Using A.M. $\geq$ G.M.

$\frac{1+a_1+a_1^2}{3} \geq a_1 \Rightarrow 1 + a_1 + a^2_1 \geq 3a_1$

similarly $1 + a_2 + a^2_2 \geq 3a_2$

: : :

$1 + a_n + a^2_n \geq 3a_n$

multiplying

$(1 + a_1 + a^2_1)\ (1 + a_2 + a^2_2)\ ....(1 + a_n + a^2_n) \geq 3^n(a_1\ a_2\ a_3 ......\ a_n)$

Section (E) :

E-1. (i) $S = 1 + 5 + 13 + 29 + 61 + ...$n terms ...(i)

$S = 1 + 5 + 13 + 29 + 61 + ...T_n$ ...(ii)

(i) – (ii) we get

$0 = 1 + [\ 4 + 8 + 16 + 32 + ...(n - 1)\ \text{term}] - T_n$

$T_n = 1 + \frac{4\ \ (2^{n-1}-1)}{(2-1)} = 1+2^{n+1} - 4 = 2^{n+1} - 3$

$S_n = \sum T_n = \sum\ (2^{n+1} - 3) = (2^{n+2} - 4) - 3n = 2^{n+2} - 3n - 4$

(ii) $S = \frac{3}{9}\ 3 + 33 + 333 + 3333 + ...$ n term.

$S = [9 + 99 + 999 + ...n\ \text{term}] = \frac{3}{9}\ [(10 - 1) + (10^2 - 1) + (10^3 - 1) + ...n\ \text{term}]$

$= \frac{3}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{3}{81}\ [10^{n+1} - 9n - 10]$

E-2. Let $S = \frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+$ ...... to n term $= \left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)$ + ... n term

$= (1 + 1 + 1 + ...n\ \text{times}) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+....n\ \ \text{term}\right)$

$= n - \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)} = n - 1 + \frac{1}{2n} = \frac{n\,.\,2^n - 2^n + 1}{2^n}$

E-3. (i) $T_n = 3^n - 2^n$; $S_n = \sum 3^n - \sum 2^n$ $S_n = \frac{3^{n+1}-3}{2} - \frac{2^{n+1}-2}{1}$; $S_n = \frac{1}{2}\left(3^{n+1}+1\right) - 2^{n+1}$

(ii) $\sum_{n=2}^{k} t_n = \sum_{n=2}^{k} n(n+2) = \frac{1}{6}k(k+1)(2k+7)$

(iii) clearly n^6 term of the given series is negative or positive accordingly as n is even or odd respectively

(a) n is even

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + (n-1)^2 - n^2 = (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + ...((n-1)^2 - n^2)$

$= (1-2)(1-2) + (3-4)(3+4) + (5-6)(5+6) + ... + ((n-1)-(n))(n-1+n) = -\frac{n(n+1)}{2}$

(b) n is odd $(1^2 - 2^2) + (3^2 - 4^2) + ... \{(n-2)^2 - (n-1)^2\} + n^2$

$= (1-2)(1+2) + (3-4)(3+4) + ... + [(n-2)-(n-1)][(n-2)+(n-1)] + n^2$

$= -(1+2+3+4+.......... + (n-2) + (n-1)) + n^2 = \frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2}$

(iv) $\sum_{r=1}^{10}(3r+7)^2 = 6265$ (v) $S_n = \sum_{r=1}^{n} I(r) = n(2n^2 + 9n + 13) \Rightarrow I(r) = S_r - S_{r-1}$

$= r(2r^2 + 9r + 13) - (r-1)(2(r-1)^2 + 9(r-1) + 13) = 6r^2 + 12r + 6 - 6(r+1)^2 \Rightarrow \sqrt{I(r)} = \sqrt{6}(x+1)$

$\Rightarrow \sum_{r=1}^{n}I(r) = \sqrt{6}\sum_{r=1}^{n}(r+1) = \sqrt{6}\left(\frac{n^2+3n}{2}\right) = \sqrt{\frac{3}{2}}(n^2+3n)$

E-4. (i) $S = \frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + ...$n terms; $T_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$

$\therefore$ $T_n = \frac{1}{4}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$; $T_1 = \frac{1}{4}\left[\frac{1}{1.3} - \frac{1}{3.5}\right]$, $T_2 = \frac{1}{4}\left[\frac{1}{3.5} - \frac{1}{5.7}\right]$,

$T_3 = \frac{1}{4}\left[\frac{1}{5.7} - \frac{1}{7.9}\right] \vdots$ $T_n = \frac{1}{4}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right]$ sum of all terms gives S_n

$\Rightarrow$ $S_n = \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$

(ii) $1.3.2^2 + 2.4.3^2 + 3.5.4^2 + ...$ n terms

$T_n = n(n+2)(n+1)^2 = n(n+1)(n+2)(n+3-2)$

$T_n = n(n+1)(n+2)(n+3) - 2(n)(n+1)(n+2)$

$S_n = S_1 - 2S_2$

$S_1 = \sum_{r=1}^{n} r(r+1)(r+2)(r+3) = \sum_{r=1}^{n}\frac{1}{5}[r(r+1)(r+2)(r+3)(r+4) - (r-1)r(r+1)(r+2)(r+3)$

$= \frac{1}{5}[1.2.3.4.5 - 0] + \frac{1}{5}[2.3.4.5.6 - 1.2.3.4.5] + \frac{1}{5}[3.4.5.6.7 - 2.3.4.5.6] + ... + \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3)]$

$\therefore$ $S_1 = \frac{1}{5}[n(n+1)(n+2)(n+3)(n+4)]$

Now $S_2 = \sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}\sum_{r=1}^{n}[r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)]$

$= \frac{1}{4}[1.2.3.4 - 0] + \frac{1}{4}[2.3.4.5 - 1.2.3.4] + \frac{1}{4}[3.4.5.6 - 2.3.4.5]$

$......... + \frac{1}{4}[n(n+1)(n+2)(n+3) - (n-1)n(n+1)(n+2)]$

$\therefore$ $S_2 = \frac{1}{4}[n(n+1)(n+2)(n+3)]$

$\therefore$ $S_n = \left[\frac{n(n+1)(n+2)(n+3)(n+4)}{5}\right] - \frac{2}{4}[n(n+1)(n+2)(n+3)]$

$= n(n+1)(n+2)(n+3)\left[\frac{n+4}{5} - \frac{1}{2}\right] = \frac{1}{10}n(n+1)(n+2)(n+3)(2n+3)$

PART - II

Section (A) :

A-1. $S = \frac{2p+1}{2}\,[2(p^2 + 1) + 2p] = (2p + 1)\,(p^2 + 1 + p) = 2p^3 + 3p^2 + 3p + 1 = p^3 + (p + 1)^3$

A-2. $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225 \quad \Rightarrow \quad 3\,(a_1 + a_{24}) = 225$

(sum of terms equidistant from beginning and end are equal) $a_1 + a_{24} = 75$

Now $a_1 + a_2 + \ldots\ldots + a_{23} + a_{24} = \frac{24}{2}\,[a_1 + a_{24}] = 12 \times 75 = 900$

A-3. 2, 5, 8 ....

$a = 2, d = 3 \quad \Rightarrow \quad S_{2n} = n\,(4 + (2n - 1)\,3] = n(6n + 1) \Rightarrow \quad 57, 59, 61, \ldots\ldots$

$S_n = [2 \times 57 + (n - 1)\,2] = n[57 + n - 1] = n\,(56 + n)$

$n\,(6n + 1) = n\,(56 + n) \quad \Rightarrow \quad 5n = 55 \quad \Rightarrow \quad n = 11.$

A-4. Sum of the integer divided by 2 = $2 + 4 + \ldots\ldots\ldots + 98 + 100 = \frac{50}{2}\,[2.2 + (50 - 1)2] = 50\,[51] = 2550$

Sum of the integer divided by 5 = $5 + 10 + \ldots\ldots\ldots\ldots\ldots\ldots 95 + 100 = \frac{20}{2}\,[5 + 100] = 1050$

Sum of the integer divided by 10 $\Rightarrow \frac{10}{2}\,[10 + 100] = 550$

Sum of the integers divided by 5 or 10 = 2550 + 1050 – 550 = 3050

A-5. $A_1^2 - A_2^2 + A_3^2 - A_4^2 + A_5^2 - A_6^2 = -\,d\,(A_1 + A_2 + \ldots..+ A_6) = -\left(\frac{b-a}{7}\right)(3(b + a)) = 3\left(\frac{a^2-b^2}{7}\right)$ = Prime

$\Rightarrow a = 4, b = 3$

Section (B) :

B-1. $T_3 = 4$

$T_1 . T_2 . T_3 . T_4 . T_5 = a^5 . r^{1+2+3+4} = a^5 . r^{10} = (ar^2)^5 = 4^5$

B-2. $S = \frac{a}{1-r} \Rightarrow r = \frac{S-a}{S}$; $S' = \frac{a[1-r^n]}{1-r} = S\left[1-\left(\frac{S-a}{S}\right)^n\right]$

B-3. $a_1 = 2;\ a_{n+1} = \frac{a_n}{3};\ a_2 = \frac{a_1}{3} = \frac{2}{3};\ a_3 = \frac{a_2}{3} = \frac{2}{3^2}$

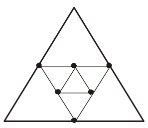
$a_1 + a_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{20} = 2 + \frac{2}{3} + \frac{2}{3^2} + \ldots\ldots\ldots = \frac{2\,.\left[1-\left(\frac{1}{3}\right)^{20}\right]}{1-\frac{1}{3}} = 3\left(1-\frac{1}{3^{20}}\right)$

B-4. $\alpha + \beta = 3$, $\alpha\beta = a$; $\gamma + \delta = 12$, $\gamma\delta = b$

$\alpha, \beta, \gamma, \delta$ are in G.P. Let r be the common ratio so $\alpha\,(1+ r) = 3$

$\alpha r^2(1+ r) = 12 \quad \Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$

so $\alpha = 1 \quad \Rightarrow \quad$ so $a = 2$, $b = 32$ **Ans**



B-5. $= 3[24 + 12 + 6 + \ldots\ldots\ldots\infty] = 3\,\frac{24}{1-\frac{1}{2}} = 144$

B-6. If a, G_1, G_2, G_3, b are in G.P. with common ratio equal to 'r' then $G_1 - a$, $G_2 - G_1$, $G_3 - G_2$, $b - G_3$ are also in G.P. with same common ratio $\Rightarrow \frac{G_3 - G_2}{G_2 - G_1} = r = 2 \Rightarrow \frac{b}{a} = r^4 = 16$

Section (C) :

C-1. $T_3 = \frac{1}{3}$, $T_6 = \frac{1}{5}$. $T_n = \frac{3}{203}$

then 3rd , 6th term of A.P. series are 3, 6, $\frac{203}{3}$

$a + 2d = 3 \Rightarrow a = 5d = 5$

$d = \frac{2}{3}$, $a = \frac{5}{3}$

$a + (n-1)d = \frac{203}{3} \Rightarrow \frac{5}{3} + (n-1) = \frac{203}{3}$

$(n-1)^2 = 198$

$n = 100$

C-2. a, b, c are in H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. $S = \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}} + \frac{\frac{1}{c}+\frac{1}{b}}{\frac{1}{c}-\frac{1}{b}}$

Let $\frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{c} = d$

$S = \frac{\left(\frac{1}{a}+\frac{1}{b}\right) - \left(\frac{1}{c}+\frac{1}{b}\right)}{d} = \frac{\left(\frac{1}{a}-\frac{1}{c}\right)}{d} = \frac{2d}{d} = 2$

C-3. $x^3 - 11x^2 + 36x - 36 = 0$

if roots are in H.P. , then roots of new equation

$\frac{1}{x^3} - \frac{11}{x^2} + \frac{36}{x} - 36 = 0$ are in A.P.

$36x^3 + 36x^2 - 11x + 1 = 0$

$36x^3 - 36x^2 + 11x - 1 = 0$

Let the roots be α, β, γ

$\alpha + \beta + \gamma = 1$

$3\beta = 1$ $(2\beta, = \alpha + \gamma)$ $\beta = \frac{1}{3}$

so middle roots in 3.

C-4. a, b, c, d $\rightarrow$; $\frac{a}{abcd}, \frac{b}{abcd}, \frac{c}{abcd}, \frac{d}{abcd} \rightarrow$

$\frac{1}{bcd}, \frac{1}{acd}, \frac{1}{abd}, \frac{1}{abc} \rightarrow$; $\frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd} \rightarrow$

abc, abd, acd, bcd $\rightarrow$

C-5. $3 + \frac{1}{4}(3+d) + \frac{1}{4^2}(3+2d) + ...+ \infty = 8$

$S = 3 + (3+d) + (3+2d) + ...+ \infty$ ...... (i)

$\frac{1}{4} S = \frac{3}{4} + \frac{1}{4^2}(3+d) + ...\infty$ ...... (ii)

(i) – (ii) we get

$\frac{3}{4} S = 3 + \frac{1}{4} d + \frac{1}{4^2} d + ...\infty$; $\frac{3}{4} S = 3 + \frac{\frac{1}{4}d}{1 - \frac{1}{4}}$

$\frac{3}{4}S = 3 + \frac{d}{3}$; $S = \frac{12}{3} + \frac{4}{9}d = 8 = 4 + \frac{4}{9}d = 8 \Rightarrow \frac{4}{9}d = 4 \Rightarrow d = 9$ **Ans**

C-6. $n\left(\frac{a+b}{2}\right) = n\left(\frac{a+b}{2ab}\right) \Rightarrow ab = 1$

C-7. If first and last term of A.P. and H.P. are same the product of x terms begining in A.P. and kth term from end in H.P. is constant and equal = first term × last term
$a_7 h_{24} + a_{14} h_{17} = ab + ab = 2ab = 2(25)(2) = 100$

C-8. Let a, b, c in G.P. then $b^2 = ac$ then $a + b, 2b, b + c$ in HP
$\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ in AP $\frac{2}{2b} = \frac{1}{a+b} + \frac{1}{b+c}$
$(a + b)(b + c) = (a + c + 2b) b \Rightarrow ab + b^2 + ac + bc = ab + bc + 2b^2$
$\therefore$ $b^2 = ac$
So statement (1) and (2) is true

C-9. $S = \frac{3^{10}}{\left(1-\frac{1}{6}\right)} + \frac{3^{10}\left(\frac{1}{6}\right)}{\left(1-\frac{1}{6}\right)^2} = \frac{6^2.3^{10}}{5^2} \Rightarrow \left(\frac{25}{36}\right)S = 3^{10}$

C-10. $S = \frac{3}{2} + \frac{15}{2^2} + \frac{35}{2^3} + \frac{63}{2^4} + ...\infty$

$\frac{1}{2}S = \frac{3}{2^2} + \frac{15}{2^3} + \frac{35}{24} + ...\infty$

$\frac{S}{2} = \frac{3}{2^2} + \frac{12}{2^2} + \frac{20}{2^3} + ...\infty$
again use same concept S = 23

Section (D) :

D-1. $x \in R$
$5^{1+x} + 5^{1-x}$, a/2, $5^{2x} + 5^{-2x}$ are in A.P
$a = (5^{2x} + 5^{-2x}) + (5^{1+x} + 5^{1-x}) \Rightarrow a = (5^{2x} + 5^{-2x}) + 5(5^x + 5^{-x}) = (5^x - 5^{-x})^2 + 2 + 5(5^{x/2} - 5^{-x/2})^2 + 10$
$a = 12 + (5^x - 5^{-x})^2 + 5(5^{x/2} - 5^{-x/2})^2 \Rightarrow a \geq 12$

D-2. AM = A = $\frac{a+b+c}{3}$; GM = G = $(abc)^{1/3}$

HM = H = $\frac{3abc}{ab+bc+ca} = \frac{3G^3}{ab+bc+ca}$.
Equation whose roots are a,b,c $\Rightarrow x^3 - (a + b + c)x^2 + (\Sigma ab)x - abc = 0$
$\Rightarrow x^3 - 3Ax^2 + \frac{3G^3}{H}.x - G^3 = 0$ **Ans**

D-3. $a + b + c + d = 2 \Rightarrow a, b, c, d > 0$
$\frac{(a+b)+(c+d)}{2} \geq \sqrt{(a+b)\ (c+d)} \Rightarrow 1 \geq \sqrt{(a+b)\ (c+d)} \geq 0$
$\Rightarrow 0 \leq (a + b)(c + d) \leq 1 \Rightarrow 0 \leq M \leq 1$

D-4. Taking A.M. and G.M. of number, $\frac{a}{2}, \frac{a}{2}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}$

we get A.M. ≥ G.M. $\dfrac{2.\dfrac{a}{2}+3.\dfrac{b}{3}+2.\dfrac{c}{2}}{7} \geq \left(\left(\dfrac{a}{2}\right)^2\left(\dfrac{b}{3}\right)^3\left(\dfrac{c}{2}\right)^2\right)^{1/7}$

or $\dfrac{3}{7} \geq \left(\dfrac{a^2b^3c^2}{2^2.3^3.2^2}\right)^{1/7}$ or $\dfrac{3^7}{7^7} \geq \dfrac{a^2b^3c^2}{2^4.3^3}$ or $a^2b^3c^2 \leq \dfrac{3^{10}.2^4}{7^7}$

$\therefore$ Greatest value of $a^2\,b^3\,c^2 = \dfrac{3^{10}.2^4}{7^7}$

D-5. $P = \dfrac{a+b}{2}$, $Q = \sqrt{ab}$; $P - Q = \dfrac{a+b-2\sqrt{ab}}{2} = \dfrac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}$

Section (E) :

E-1 $H_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{n}$

$1 + \dfrac{3}{2} + \dfrac{5}{3} + ... + \dfrac{2n-1}{n} = (2-1) + \left(2-\dfrac{1}{2}\right) + \left(2-\dfrac{1}{3}\right) + ... + \left(2-\dfrac{1}{n}\right) = 2n - H_n$

E-2. $S = 1 + 2 + 4 + 7 + 11 + 16...............T_n$ ...(i)

$S = 1 + 2 + 4 + 7 + 11 ....T_n$ ...(ii)

(i) – (ii) we get

$O = 1+(1+2+3+4+5............(n-1) \text{ term}) - T_n \Rightarrow T_n = 1 + \dfrac{(n-1)\ n}{2} = \dfrac{n^2}{2} - \dfrac{n}{2} + 1$

$\therefore$ General term $= T_n = an^2 + bn + c$ here a = 1/2, b= –1/2, c = 1

$S_n = \sum T_n = \dfrac{n(n+1)\ (2n+1)}{12} - \dfrac{n(n+1)}{4} + n$

$S_{30} = \dfrac{30.\ 31.\ 61}{12} - \dfrac{30.\ 31}{4} + 30 = 4727.5 - 232.5 + 30 = 4525$ **Ans (D)**

E-3. $\sum\limits_{r=1}^{n} \dfrac{1}{\sqrt{a+r\ x}+\sqrt{a+(r-1)\ x}}$; $\sum\limits_{r=1}^{n} \dfrac{\sqrt{(a+rx)}-\sqrt{a+(r-1)\ x}}{(a+rx)-(a+(r-1)\ x)}$;

$= \dfrac{1}{x}\left[(\sqrt{a+x}-\sqrt{a+0.x})+(\sqrt{a+2x}-\sqrt{a+x})+(\sqrt{a+3x}-\sqrt{a+2x}) + ... + (\sqrt{a+nx}-\sqrt{a+(n-1)\ x})\right]$

$= \dfrac{1}{x}\left[\sqrt{a+nx}-\sqrt{a}\right] = \dfrac{n}{\sqrt{a}+\sqrt{a+nx}}$ **Ans**

E-4. $\sum\limits_{r=1}^{10}(2r-1)r^2 = \sum\limits_{r=1}^{10} 2r^3 - \sum\limits_{r=1}^{10} r^2 = 6050 - 385 = 5665$

PART - III

1. **(A)** $1x^{50} - (1+3+5+......+99)x^{49} + (\quad)x^{48}$ .... $\Rightarrow$ coefficient $x^{49} = -50^2 = -2500$

(B) $\dfrac{S_{2n}}{S_n} = \dfrac{\dfrac{2n}{2}\left[2a+(2n-1)d\right]}{\dfrac{n}{2}\left[2a+(n-1)d\right]} = 3 = \dfrac{2a+(2n-1)d}{2a+(n-1)d} = \dfrac{3}{2}$ $\therefore$ $d = \dfrac{2a}{n+1}$

Now $\dfrac{S_{3n}}{S_n} = \dfrac{\dfrac{3n}{2}\left[2a+(3n-1)d\right]}{\dfrac{n}{2}\left[2a+(n-1)d\right]} = \dfrac{3\left[2a+(3n-1)\dfrac{2a}{n+1}\right]}{\left[2a+(n-1)\dfrac{2a}{n+1}\right]} = \dfrac{3\left[(n+1)+(3n-1)\right]}{(n+1)+(n-1)} = \dfrac{3.4n}{2n} = 6$

(C) $S = \sum\limits_{r=2}^{\infty}\left(\dfrac{1}{r^2-1}\right)$; $S = \dfrac{1}{2}\sum\limits_{r=2}^{\infty}\left(\dfrac{1}{r-1}-\dfrac{1}{r+1}\right)$

$S = \dfrac{1}{2}\left[\left(\dfrac{1}{1}-\dfrac{1}{3}\right) + \left(\dfrac{1}{2}-\dfrac{1}{4}\right)+\left(\dfrac{1}{3}-\dfrac{1}{5}\right)+\left(\dfrac{1}{4}-\dfrac{1}{6}\right)+...............\infty\right] = \dfrac{1}{2}\left[1+\dfrac{1}{2}\right] = \dfrac{3}{4}$

(D) ℓ, ℓr, ℓr^2 $\therefore$ $\ell^3 r^3 = 27$ (volume) $\Rightarrow$ $\ell r = 3$

surface area

$2(\ell . \ell r + \ell r . \ell r^2 + \ell r^2 \ell) = 78 \quad \Rightarrow \quad \ell^2 (r + r^2 + r^3) = 39 \Rightarrow \ell^2 \left(\frac{3}{\ell} + \frac{3^2}{\ell^2} + \frac{3^3}{\ell^3}\right) = 39$

$3\ell + 3^2 + \frac{3^3}{\ell} = 39 \quad \Rightarrow \quad \ell + 3 + \frac{9}{\ell} = 13$

$\therefore \quad \ell^2 - 10\ell + 9 = 0 \quad \Rightarrow \quad \ell = 1, 9 \quad \Rightarrow \quad \ell r = 3$ and $\Rightarrow \ell > \ell r$

$\therefore \quad r = \frac{1}{3} \qquad \therefore \quad \ell = 9$

2. (A) a, x, y, z, b in A.P. & a, x, y, z, b in H.P.

$\frac{1}{b}, \frac{1}{z}, \frac{1}{y}, \frac{1}{x}, \frac{1}{a}$ in A.P. ; $a, \frac{ab}{z}, \frac{ab}{y}, \frac{ab}{x}, b$ in A.P,

$\Rightarrow \quad ab = xz = y^2 = zx = ba$

$\Rightarrow \quad xz.y^2 .zx = (ab)(ab)(ab)$

$\Rightarrow \quad (xyz)(xyz) = a^3 b^3$

$ab = 3 \quad \Rightarrow \quad a = 1, b = 3$

or $\quad a = 3, b = 1$

(B) $S_\infty = 2^{1/4} .4^{1/8} .8^{1/16}................\infty$; $S_\infty = 2^{1/4} .2^{2/8} .2^{3/16}................\infty$

$S_\infty = 2^{1/4 + 2/8 + 3/16 ....\infty} = 2^{S'_\infty}$

Let $S'_\infty = \frac{1}{4} + \frac{2}{8} + \frac{3}{16}$ ....

$S'_n = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} .... \frac{n}{2^{n+1}}$ ...(i)

$\frac{S'_n}{2} = \frac{1}{8} + \frac{2}{16} + ... + \frac{n-1}{2^{n+1}} + \frac{n}{2^{n+2}}$ ...(ii)

(i) – (ii) we get

$\frac{S'_n}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... \frac{1}{2^{n+1}} - \frac{n}{2^{n+2}}$

$\Rightarrow \quad \frac{S'_n}{2} = \frac{1}{4}\left(\frac{1-(1/2)^n}{1-1/2}\right) - \frac{n}{2^{n+2}}$

$S'_n = \frac{2.2}{4}\left(1-\left(\frac{1}{2}\right)^n\right) - \frac{2n}{2^{n+2}} \Rightarrow S'_\infty = 1$

$\therefore \quad S_\infty = 2^{S'_\infty} = 2$

(C) x,y,z are in A.P $y = \frac{x+z}{2}$,

or $2y = x + z$

$(x + 2y - z)(x + z + z - x)(z + x - y) = (x + (x + z) - z)(x + z + z - x)(2y - y) = 2x .2z .y = 4xyz$

$\therefore \quad k = 4$

(D) $d = \frac{31-1}{m+1} = \frac{30}{m+1}$; $\frac{A_7}{A_{m-1}} = \frac{1+7\frac{30}{m+1}}{1+(m-1)\frac{30}{m+1}} = \frac{5}{9}$

$\Rightarrow \frac{m+211}{31m-29} = \frac{5}{9}$

$\Rightarrow$ 146 m = 2044 $\Rightarrow$ m = 14

$\therefore \quad \frac{m}{7} = 2$

EXERCISE # 2

PART - I

1. $\frac{x_1}{x_1+1} = \frac{x_2}{x_2+3} = \frac{x_3}{x_3+5} = \ldots\ldots = \frac{x_{2013}}{x_{2013}+4025} = \frac{1}{\lambda}$

$\Rightarrow \quad x_1 = \frac{1}{\lambda-1}, x_2 = \frac{3}{\lambda-1}, x_3 = \frac{5}{\lambda-1}, \ldots\ldots, x_{2013} = \frac{4025}{\lambda-1}$

$\Rightarrow \quad x_1, x_2, x_3, \ldots\ldots, x_{2013}$ are in A.P. with common difference = $\frac{2}{\lambda-1}$ = d

$x_1, x_2, x_3, \ldots\ldots, x_{2013} = \frac{2}{\lambda-1}$ = d

2. 2b = a + c and $b^2 = \pm ac$

case-I

if $b^2 = ac$ and $a + c + b = \frac{3}{2}$ $\Rightarrow$ $b = \frac{1}{2}$

a + c = 1 $\Rightarrow$ $ac = \frac{1}{4} \Rightarrow$ $(1-c)c = \frac{1}{4}$

$c^2 - c + \frac{1}{4} = 0 \Rightarrow$ $c = \frac{1}{2}$ $\Rightarrow$ $a = \frac{1}{2}$

a = b = c so not valid

case-II

$b^2 = -ac$ and $b = \frac{1}{2}$; a + c = 1 $\Rightarrow$ $ac = -\frac{1}{4}$

$(1-c)c = -\frac{1}{4}$ $\Rightarrow$ $c^2 - c - \frac{1}{4} = 0$

$\Rightarrow$ $c = \frac{1\pm\sqrt{1+1}}{2} = \frac{1\pm\sqrt{2}}{2}$

$c = \frac{1+\sqrt{2}}{2}$ $\Rightarrow$ $a = \frac{1-\sqrt{2}}{2}$

3. $S_1 + S_2 + S_3 + \ldots\ldots + S_p$

$\Rightarrow \quad S_1 = \frac{n}{2}[2.1 + (n-1)1]$ $S_2 = \frac{n}{2}[2.2 + (n-1)3]$

$S_3 = \frac{n}{2}[2.3 + (n-1)5]$

⋮

⋮

$S_p = \frac{n}{2}[2.p + (n-1)(2p-1)]$

So $\quad S_1 + S_2 + ... + S_p = \frac{n}{2}\,[2(1 + 2 + ... p) + (n - 1)(1 + 3 + 5 + ... (2p - 1))]$

$= \frac{n}{2}\left[2\ .\ \frac{p\ \ (p+1)}{2} + (n-1)\ \ p^2\right] = \frac{n\ \ p}{2}\ \ (n\,p + 1)$ **Ans.**

4. $\frac{ar^{p-1}(r^n - 1)}{r - 1} = k\ \frac{ar^{q-1}(r^n - 1)}{r - 1}$. ; $\quad r^{p-1} = k.\ r^{q-1}$; $\quad k = r^{p-q}$

5. $45^2 = 2025 \quad \& \quad 46^2 = 2116$

$\Rightarrow$ there are 45 squares ≤ 2056

which are left only from sequence of possible integers since $2056 = 2011 + 45$

$\therefore \quad 2011^{th}$ term $= 2056$

6. $x = \frac{1}{1-a},\ y = \frac{1}{1-b},\ z = \frac{1}{1-c} \Rightarrow$ a, b, c are in A.P.

$\Rightarrow \quad 1 - a,\ 1 - b,\ 1 - c$ are also in A.P. $\Rightarrow \quad \frac{1}{1-a},\ \frac{1}{1-b},\ \frac{1}{1-c}$ are in H.P.

7. $f(k)\ \sum_{r=1}^{n}(a_r - a_k) = \sum_{r=1}^{n} a_r - \sum_{r=1}^{n} a_k \quad f(k) = \lambda - na_k\ f(i) = \lambda - na_i\ \frac{a_i}{f(i)} = \frac{a_i}{\lambda - na_i} = \frac{1}{\frac{\lambda}{a_i} - n}$ $<a_i>$ in A.P.

$\Rightarrow\ <\frac{1}{a_i}>$ in A.P. $\Rightarrow\ <\frac{\lambda}{a_i} - n>$ in A.P. $\Rightarrow\ <\frac{1}{\frac{\lambda}{a_i} - n}>$ in H.P. $\Rightarrow\ <\frac{a_i}{f(i)}>$ in A.P.

8. $a_1 \cdot a_2 \cdot a_3 \cdot ....a_n = c \quad \Rightarrow \frac{a_1 + a_2 + a_3 + ...2a_n}{n} \geq (a_1a_2a_3\ ...\ 2a_n)^{1/n} \geq (2c)^{1/n}$

$\Rightarrow \quad a_1 + a_2 + a_3 + ...2a_n \geq n(2\ c)^{1/n}$

9. $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...n$ terms $= \frac{n(n+1)^2}{2}$, when n is even

$1^2 + 2.\ 2^2 + 3^2 + ...2.\ n^2 = n\ \frac{(n+1)^2}{2} \Rightarrow$ when n is odd $\quad n + 1$ is even

$1^2 + 2.\ 2^2 + 3^2 + ...\ n^2 + 2.\ (n + 1)^2 = (n + 1)\ \frac{(n+2)^2}{2}$

$1^2 + 2.\ 2^2 + 3^2 + ...\ n^2 = (n + 1)\left[\frac{(n+2)^2}{2} - 2(n+1)\right] = \frac{(n+1)\ \ n^2}{2}$

10. $S_n - S_{n-2} = 2 \quad \Rightarrow \quad T_n + T_{n-1} = 2$

Also $T_n - T_{n-1} = 2;\ T_n + T_{n-1} = \left(\frac{1}{n^2} + 1\right) T_{n-1} = 2 \Rightarrow T_{n-1} = \frac{2}{1 + \frac{1}{n^2}} = \frac{2n^2}{1+n^2}$ So $T_m = \frac{2(m+1)^2}{1+(m+1)^2}$

11. If $1^2 + 2^2 + 3^2 + ... + 2003^2 = (2003)\ (4007)\ (334)$

$(1)\ (2003) + (2)\ (2002) + (3)\ (2001) + ... + (2003)\ (1) = (2003)\ (334)\ (x)$

$\Rightarrow \sum_{r=1}^{2003} r\ \ (2003 - r + 1) = (2003)\ (334)\ (x) \quad \Rightarrow \quad 2004.\ \sum_{r=1}^{2003} r - \sum_{r=1}^{2003} r^2 = (2003)\ (334)\ (x)$

$\Rightarrow \quad 2004\left(\frac{2003\ \ .\ \ 2004}{2}\right). - 2003.\ (4007).\ 334 = (2003)\ (334)\ (x)$

$\Rightarrow \quad x = 2005$ **Ans.**

12. $\sum_{r=1}^{n} t_r = S_n \quad \Rightarrow \quad \sum_{r=1}^{n-1} t_r = S_{n-1} \Rightarrow t_n = S_n - S_{n-1} = \frac{n(n+1)(n+2)}{2}$

$$\sum_{r=1}^{n}\frac{1}{t_r} = \sum_{r=1}^{n}\frac{2}{r(r+1)(r+2)} = \sum\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right) = -\left(\frac{1}{(n+1)(n+2)}-\frac{1}{2}\right)$$

13. $\because$ $I = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\infty = \frac{\pi^2}{6}$

Let $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = A$

$\because$ $I = \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+......\infty\right) + \frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+......\infty\right)$

$\Rightarrow$ $I = A + \frac{I}{4}$ $\Rightarrow A = \frac{3I}{4} = \frac{3}{4}\times\frac{\pi^2}{6}$ $\Rightarrow A = \frac{\pi^2}{8}$

14. $= (-1^2 + 2^2 + 3^2 + 4^2 - 5^2 + 6^2 + 7^2 + 8^2 .....) + (1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2.....) - 4(6^2 + 14^2 + ...)$
$= 2[2^2 - 6^2 + 10^2 - 14^2 + ...]$ {2n terms }
$= 2[(2-6)(2+6) + (10-14)(10+14).....]$
$= 2 \times (-4)[2 + 6 + 10 + 14 .....]$ {2n terms }
$= 2 \times (-4) \times 2[1 + 3 + 5 + 7 .....]$ {2n terms }
$= -16(2n)^2 = -64n^2 = -(8n)^2$

PART - II

1. Let first installment be = 'a' and the common difference of the A.P. be 'd'
So $a + (a + d) + (a + 2d) + ... + (a + 39d) = 3600$

$\Rightarrow$ $\frac{40}{2}[2a + 39d] = 3600$

$\Rightarrow$ $2a + 39d = 180$ ..... (1)

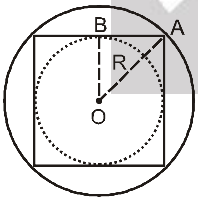
and $\frac{30}{2}[2a + 29d] = 2400$

$\Rightarrow$ $2a + 29d = 160$ ..... (2)

By equations (1) & (2), we get
$d = 2$ and $a = 51$ **Ans.**

2. Area $= A_1 = \pi R^2$ $\Rightarrow$ $OB = \frac{R}{\sqrt{2}}$

So Area $A_2 = \pi\left(\frac{R^2}{2}\right)$. So $\lim_{n\to\infty}\left((\pi R^2)+\frac{\pi R^2}{2}+\frac{\pi R^2}{4}+....\infty\right) = \pi R^2 \frac{1}{\left(1-\frac{1}{2}\right)} = 2\pi R^2$



Now sum of areas of the squares $= 2R^2 + \frac{2R^2}{2} + \frac{2R^2}{4} + ... \infty = \frac{2R^2}{1-\frac{1}{2}} = 4R^2$

3. $a + 6d = 9$; $T_1 T_2 T_7 = a(a + d)(a + 6d) = 9a(a + d) = 9(9 - 6d)(9 - 5d)$

$\because$ $T_7 = a + 6d = 9.$

Let $A = T_1 T_2 T_7$

$\frac{dA}{d(d)} = 9[-45 - 54 + 60d] = 0$ $\Rightarrow$ $60d = 99$

$\Rightarrow$ $d = \frac{33}{20}$ Ans.

4. Let AP has 2n terms

Sum of odd term = 24 $\Rightarrow$ $\frac{n}{2}[a_1 + a_{2n-1}] = 24$ ..... (1)

and sum of even terms = 30 $\Rightarrow$ $\frac{n}{2}[a_2 + a_{2n}] = 30$ ..... (2)

and $a_{2n} = a_1 + \frac{21}{2}$

$a_1 + (2n-1)d = a_1 + \frac{21}{2} \Rightarrow (2n-1)\,d = \frac{21}{2}$ ..... (3)

By equation (1) & (2)

$a_1 + a_{2n-1} = \frac{48}{n}$ and $a_2 + a_{2n} = \frac{60}{n}$

So $a_1 + (n-1)d = \frac{24}{n}$ and $a_1 + nd = \frac{30}{n}$ So $d = \frac{6}{n}$

Now $(2n-1)\,\frac{6}{n} = \frac{21}{2} \Rightarrow n = 4,\; d = \frac{6}{4} = \frac{3}{2}$

So no. of terms = 2n = 8 and $a_1 = 3/2$. Numbers are $\frac{3}{2}$, 3, $\frac{9}{2}$ ....

5. AP(1, 3) = {1, 4, 7, 10, 13, 16……}
AP(3, 5) = {3, 8, 13, 18, ……}
AP(5, 7) = {5, 12, 19, 26, 33, ……}
AP(1, 3) $\cap$ AP(3, 5) = AP(13, 15) = {13, 28, 43, 58, 73, 88, 103, ……}
AP(1, 3) $\cap$ AP(3, 5) $\cap$ AP(5, 7) = AP{103, 105)

6. $\log_2 x + \log_2(\sqrt{x}) + \log_2(x)^{1/4} + \log_2(x)^{1/8} + ... = 4 \Rightarrow \log_2 x \frac{1}{2} + \log_2 x + \frac{1}{4}\log_2 x + ... = 4$

$\Rightarrow \frac{\log_2 x}{1-\frac{1}{2}} = 4 \Rightarrow \log_2 x = 2 \Rightarrow x = 4$

$$\frac{\left(\frac{x^2+3x+2}{x+2}\right) + 3x - \frac{x(x^3+1)}{(x+1)(x^2-x+1)}\log_2 8}{(x-1)(\log_2 3)(\log_3 4)(\log_4 5)(\log_5 2)} = \frac{x+1}{x-1}$$

7. $\alpha + \gamma = \frac{4}{A},\ \alpha\gamma = \frac{1}{A}$ and $\beta + \delta = \frac{6}{B},\ \beta\delta = \frac{1}{B}$

Since $\alpha, \beta, \gamma, \delta$ are in H.P. $\beta = \frac{2\alpha\gamma}{\alpha+\gamma} = \frac{1}{2}$ is root of $Bx^2 - 6x + 1 = 0 \Rightarrow B = 8$

similarly $\gamma = \frac{2\beta\delta}{\beta+\delta} = \frac{1}{3}$ is root of $Ax^2 - 4x + 1 = 0 \Rightarrow A = 3$

8. $ar^2, 3a.ar^3, ar$ are in A.P. d = 1/8
$3a^2r^3 - ar^2 = 1/8$ .....(1) ; $ar = a^2 + 2.1/8$ ......(2)

from (2) a = $(-)^2 = \frac{1}{4r(1-r)}$ ......(3)

from (1) & (3) $r = \frac{1}{2}$; r = –2 but $0 < r < 1$

$r = \frac{1}{2} \Rightarrow a = 1 \Rightarrow 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ .... sum = $\frac{1}{1-\frac{1}{2}} = 2$

9. a, b, c are in G.P. $\Rightarrow$ $b^2 = ac$ $\Rightarrow$ (a – b), (c – a), (b – c) are in H.P.

So $\frac{1}{a-b}, \frac{1}{c-a}, \frac{1}{b-c}$ are in AP. Let a, b, c are $\frac{b}{r}$, b, br

So $\frac{1}{\frac{b}{r}-b}, \frac{1}{br-\frac{b}{r}}, \frac{1}{b-br}$ are in AP. So $\frac{2}{br-\frac{b}{r}} = \frac{1}{\frac{b}{r}-b} + \frac{1}{b-br}$

$\Rightarrow \frac{2r}{r^2-1} = \frac{r}{1-r} + \frac{1}{1-r}$ $\Rightarrow$ $-\frac{2r}{(r+1)} = (1+r)$ $\Rightarrow$ $(1+r)^2 = -2r$

$\Rightarrow r^2 + 1 + 4r = 0$ $\Rightarrow$ $\frac{c}{a} + 1 + \frac{4b}{a} = 0$ $\Rightarrow$ $a + 4b + c = 0$

10. $a, a_1, a_2, \ldots\ldots, a_{2n}$, b are in AP and $a, g_1, g_2, \ldots\ldots g_{2n}$, b are in GP and $h = \frac{2ab}{a+b}$

$\therefore \frac{a_1+a_{2n}}{g_1 \; g_{2n}} + \frac{a_2+a_{2n-1}}{g_2 \; g_{2n-1}} + \ldots\ldots + \frac{a_n+a_{n+1}}{g_n \; g_{n+1}} = \frac{a+b}{ab} + \frac{a+b}{ab} + \ldots\ldots + \frac{a+b}{ab} = 2n\left(\frac{a+b}{2ab}\right) = \frac{2n}{h}$

11. $\frac{a+b}{2} = 6$ $G^2 + 3H = 48$ $\Rightarrow$ $ab + 3\frac{2ab}{a+b} = 48$ $\Rightarrow$ $ab + \frac{3ab}{6} = 48$

$\Rightarrow \frac{3}{2}ab = 48$ $\Rightarrow$ $ab = 32$ $\Rightarrow$ $a = 4$, $b = 8$.

12. $S = \frac{5}{13} + \frac{55}{(13)^2} + \frac{555}{(13)^3} + \ldots\ldots\ldots\infty$; $S = \frac{5}{9}\left[\frac{(10-1)}{13} + \frac{10^2-1}{(13)^2} + \frac{10^3-1}{(13)^3} + \ldots\ldots\infty\right]$

$= \frac{5}{9}\left[\frac{\frac{10}{13}}{1-\frac{10}{13}} - \left(\frac{\frac{1}{13}}{1-\frac{1}{13}}\right)\right] = \frac{5}{9}\left[\frac{10}{3} - \frac{1}{12}\right] = \frac{5}{9}\left[\frac{39}{12}\right] = \frac{65}{36}$ **Ans**

13. $S = 1^2 - \frac{2^2}{5} + \frac{3^2}{5^2} - \frac{4^2}{5^3} + \frac{5^2}{5^4} - \frac{6^2}{5^5} + \ldots\ldots\ldots\ldots\infty$ ...(i)

$-\frac{1}{5}S = \frac{1}{5} - \frac{2^2}{5^2} + \frac{3^2}{5^3} - \frac{4^2}{5^4} + \frac{5^2}{5^5} - + \ldots\ldots\ldots\ldots\infty$ ...(ii)

(i) – (ii) we get

$\frac{6}{5}S = 1 - \frac{3}{5} + \frac{5}{5^2} - \frac{7}{5^3} + \frac{9}{5^4} - \frac{11}{5^5} + \ldots\ldots\ldots\ldots\ldots\infty$ ...(iii)

$-\frac{6}{25}S = -\frac{1}{5} + \frac{3}{5^2} - \frac{5}{5^3} + \frac{7}{5^4} - \frac{9}{5^5} + \ldots\ldots\ldots\ldots\ldots\infty$ ...(iv)

(iii) – (iv) we get

$\frac{6s}{5}\left(\frac{6}{5}\right) = 1 - \frac{2}{5} + \frac{2}{5^2} - \frac{2}{5^3} + \frac{2}{5^4} - \ldots\ldots\ldots\infty$

$\frac{36}{25}S = 1 - \frac{2}{5}\left[\frac{1}{1+\frac{1}{5}}\right]$; $\frac{36}{25}S = 1 - \frac{2}{5}\left(\frac{5}{6}\right) = \frac{2}{3}$; $S = \frac{25}{36} \times \frac{2}{3} = \frac{25}{54}$ **Ans**

14. $x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots\ldots + x_{50} = 50$

AM $\geq$ HM

$\frac{x_1 + x_2 + \ldots\ldots\ldots + x_{50}}{50} \geq \frac{1}{\frac{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_{50}}}{50}}$

$\Rightarrow \dfrac{x_1 + x_2 + ...+ x_{50}}{50} \geq \dfrac{50}{\dfrac{1}{x_1}+\dfrac{1}{x_2}+.......+\dfrac{1}{x_{50}}}$

$\Rightarrow \dfrac{1}{x_1}+\dfrac{1}{x_2}+.......+\dfrac{1}{x_{50}} \geq 50$

so min value of $\dfrac{1}{x_1}+\dfrac{1}{x_2}+.......+\dfrac{1}{x_{50}} = 50$

15. $\dfrac{(a_1+a_2)+(a_3+a_4)}{2} \geq \sqrt{(a_1+a_2)(a_3+a_4)}$

$\Rightarrow (a_1+a_2)(a_3+a_4) \leq \dfrac{225}{4}$

16. $\dfrac{S_3(1+8\ \ S_1)}{S_2^2} = \dfrac{\left[\dfrac{n\ \ (n+1)}{2}\right]^2\left[1+\dfrac{8n(n+1)}{2}\right]}{\left[\dfrac{n(n+1)\ \ (2n+1)}{6}\right]^2} = \dfrac{[1+4n\ \ (n+1)]\ \ 9}{(2n+1)^2} = 9$ **Ans**

17. $T_n = \dfrac{n}{1+n^2+n^4} = \dfrac{1}{2}\left[\dfrac{(2n)}{(1+n+n^2)(1-n+n^2)}\right]$; $T_n = \dfrac{1}{2}\left[\dfrac{1}{1-n+n^2}-\dfrac{1}{1+n+n^2}\right]$

$T_1 = \dfrac{1}{2}\left[\dfrac{1}{1}-\dfrac{1}{3}\right]$, ; $T_2 = \dfrac{1}{2}\left[\dfrac{1}{3}-\dfrac{1}{7}\right]$, $T_3 = \dfrac{1}{2}\left[\dfrac{1}{7}-\dfrac{1}{13}\right]$,

$\vdots$

$T_n = \dfrac{1}{2}\left[\dfrac{1}{1-n+n^2}-\dfrac{1}{1+n+n^2}\right]$

$S_n = \sum T_n = \dfrac{1}{2}\left[1-\dfrac{1}{1+n+n^2}\right] = \dfrac{n+n^2}{2(1+n+n^2)}$

PART - III

1. Let a, a + d, a + 2d .... are Interior angles

$\because$ sum of interior angles = (n – 2) π , where n is the number of sides

$\therefore$ a = 120°, d = 5° $\Rightarrow$ $\dfrac{n}{2}$ [240° + (n – 1) 5°] = (n – 2) 180°

$\Rightarrow$ $n^2 = 25n - 144$ $\Rightarrow$ n = 16, 9 but $n \neq 16$

because if n = 16, then an interior angle will be 180° which is not possible. So n = 9

2. 1, $\log_y x$, $\log_z y$, $-15 \log_x z$ are in AP. Let commen diff. is d.

$\log_y x = 1 + d \Rightarrow x = (y)^{1+d}$; $\log_z y = 1 + 2d \Rightarrow y = (z)^{1+2d}$

$-15 \log_x z = 1 + 3d \Rightarrow z = x^{\left(\frac{1+3d}{-15}\right)}$

So $x = (y)^{1+d} = ((z)^{1+2d})^{1+d} \Rightarrow x = (x)^{\left(\frac{1+3d}{-15}\right)(1+d)(1+2d)}$

So (1 + d) (1 + 2d) (1 + 3d) = –15

So $d = -2 \Rightarrow x = (y)^{-1} \Rightarrow y = (z)^{-3} \Rightarrow z = (x)^{1/3} \Rightarrow z^3 = x$. **Ans.**

3. (D) $a_1 + 4a_2 + 6a_3 - 4a_4 + a_5 = 0 \Rightarrow a - 4(a + d) + 6(a + 2d) - 4(a + 3d) + (a + 4d) = 0 - 0 = 0$

Like wise we can check other options

4. $\dfrac{1}{16}$ a, b are in G.P. hence $a^2 = \dfrac{b}{16}$ or $16a^2 = b$ ......(1)

a, b, $\frac{1}{6}$ are in H.P. hence, $b = \frac{2a\frac{1}{6}}{a+\frac{1}{6}} = \frac{2a}{6a+1}$ ....(2)

From (1) and (2)

$16a^2 = \frac{2a}{6a+1} \Rightarrow$ $2a = \left(8a - \frac{1}{6a+1}\right) = 0 \Rightarrow$ $48a^2 + 8a - 1 = 0$ (a 0)

$\Rightarrow$ $(4a + 1)(12a - 1) = 0$ $\therefore$ $a = -\frac{1}{4}, \frac{1}{12}$

when $a = \frac{-1}{4}$, then from (1) ; $b = 16\left(-\frac{1}{4}\right)^2 = 1 \Rightarrow$ when $a = \frac{1}{12}$ then from (1) $\Rightarrow b = 16\left(\frac{1}{12}\right)^2 = \frac{1}{9}$

therefore $a = -\frac{1}{4}$, $b = 1$ or $a = \frac{1}{12}$, $b = \frac{1}{9}$

5. We have 1111.....1 (91 digits) = $10^{90} + 10^{89} + ... + 10^2 + 10^1 + 10^0$

$= \frac{10^{91}-1}{10-1} = \frac{(10^{91}-1)}{10-1} \times \left(\frac{10^7-1}{10^7-1}\right) = \frac{10^{91}-1}{10^7-1}\left(\frac{10^7-1}{10-1}\right)$

$= (10^{84} + 10^{77} + 10^{70} + ... + 1)(10^6 + 10^5 + ... + 1)$

Thus 111.... 1 (91 digits) is not a prime number

6. $a + b + c = 25 \Rightarrow$ $2a = 2 + b$ $\Rightarrow$ $c^2 = 18b$ $\Rightarrow \frac{1}{2}\left(2+\frac{c^2}{18}\right)+\frac{c^2}{18}+c = 25$

$\Rightarrow$ $c = 12, -24 \Rightarrow$ $c \neq -24$ $\Rightarrow$ $b = \frac{c^2}{18} = 8$ $\Rightarrow$ $a = 5$

7. $\because \frac{a}{1-r} = 4$ and $ar = \frac{3}{4}$

$\therefore$ $\frac{3/4r}{1-r} = 4$ $\Rightarrow$ $\frac{3}{4r} = 4 - 4r$

$16r^2 - 16r + 3 = 0$ $\Rightarrow$ $16r^2 - 12r - 4r + 3 = 0$

$4r(4r - 3) - 1(4r - 3) = 0$ $\Rightarrow$ $(4r - 3)(4r - 1) = 0$ $\Rightarrow$ $r = \frac{1}{4}, \frac{3}{4}$

8. $\left(1+\frac{1}{\sqrt{2}}+\frac{1}{2}+.....\right)$ + $(1+(\sqrt{2})+(2\sqrt{2}-1)+......)$ $\Rightarrow$ $r = 1/\sqrt{2}$ $d = \sqrt{2} - 1$

9. $a_{k-1}[a_{k-2} + ak] = 2a_k a_{k-2}$

$a_{k-1} = \frac{2a_k a_{k-2}}{a_k + a_{k-2}} \Rightarrow \frac{1}{a_{k-2}}, \frac{1}{a_{k-1}}, \frac{1}{a_k}$ are in A.P. $\Rightarrow$ A_{k-2}, A_{k-1}, A_k are in A.P.

Now $\frac{S_{2p}}{S_p} = \frac{\frac{2p}{2}(2a+(2p-1)d)}{\frac{p}{2}(2a+(p-1)d)} \Rightarrow$ for independent of p

$2a - d = 0$ $\Rightarrow$ $d = 2a$ or $d = 0$

if $d = 2a$ $\Rightarrow$ $A_1 = 1, d = 2$

$A_{2016} = 1 + 2015 \times 2 = 4031$

$\Rightarrow$ $a_{2016} = \frac{1}{A_{2016}} = \frac{1}{4031}$

If $d = 0$, $A_1 = 1 = A_2 = A_3 .... = A_{2016}$

10. $\frac{a_{k+1}}{a_k}$ is constant $\therefore$ G.P.

$a_n > a_m$ for $n > m$ $\therefore$ increasing G.P.

$a_1 + a_n = 66$ $\qquad$ $a_2 a_{n-1} = 128$

$a + ar^{n-1} = 66$ $\qquad$ $a.ar^{n-1} = 128$ .... (2)

$a(1 + r^{n-1}) = 66$ .... (1)

$a(66 - a) = 128 \Rightarrow a^2 - 66a + 128 = 0$

$(a - 2)(a - 64) = 0 \Rightarrow a = 2, a = 64$

$\therefore \quad r^{n-1} = 32$

$$\sum_{i=1}^{n} a_i = 126 \Rightarrow \frac{a(r^n - 1)}{r - 1} = 126 \Rightarrow \frac{2(32r - 1)}{r - 1} = 126$$

$\Rightarrow \quad 64r = 126 + 124 \Rightarrow n = 6$

11. **Case - I**

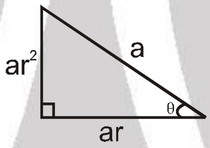
$r > 1$

$a^2 + a^2r^2 = a^2r^4 \quad \Rightarrow \quad r^4 - r^2 - 1 = 0$

$r^2 = \frac{\sqrt{5}+1}{2}$; $r = \sqrt{\frac{\sqrt{5}+1}{2}}$

tangent of smallest angle = $\tan\theta = \frac{1}{r} = \sqrt{\left(\frac{2}{\sqrt{5}+1}\right)}$

Case - II



$0 < r < 1$

$a^2 = a^2r^2 + a^2r^4$

$\Rightarrow \quad r^4 + r^2 - 1 = 0$

$r^2 = \frac{\sqrt{5}-1}{2}$; $r = \sqrt{\frac{\sqrt{5}-1}{2}}$

tangent of smallest angle = $\tan\theta = r = \sqrt{\frac{\sqrt{5}-1}{2}}$

12. b_1, b_2, b_3 are in G.P.

$\therefore \quad b_3 > 4b_2 - 3b_1$

$\Rightarrow \quad r^2 > 4r - 3$

$\Rightarrow \quad r^2 - 4r + 3 > 0$

$\Rightarrow \quad (r - 1)(r - 3) > 0$

So $\quad 0 < r < 1$ and $r > 3$

13. Let a = 1, then $s_1 = 2017$. If a 1 then $s = \frac{a^{2017} - 1}{a - 1}$.

but $a^{2017} = 2a - 1$, therefore, $S_2 = \frac{2(a-1)}{a-1} = 2$

14. H.P. is 10, 12, 15, 20, 30, 60

a = 15, b = 30, c = 60

A.P. is 15, 20, 25, ….., 55, 60

sum of all term of A.P. is 10/2 (15 + 60) = 375

15. $2x = a + b$ ..... (1)

$y^2 = ab$ ..... (2)

$z = \frac{2\ ab}{a+b}$ ..... (3)

$x = y + 2$ ..... (4)

and $\quad a = 5z$ ..... (5)

$z = \frac{2\ y^2}{2x} \quad \Rightarrow \quad y^2 = xz$

$\because$ $x = y + 2$

$\therefore$ $\frac{a+b}{2} = \sqrt{ab} + 2$..... (6)

and $a = 5\frac{(2ab)}{a+b}$

$\Rightarrow$ $(a + b) = 10\,b$

$\Rightarrow$ $a = 9b$ ..... (7)

$\frac{9b+b}{2} = \sqrt{b\ .\ 9b} + 2$

$\Rightarrow$ $5b = 3b + 2$

$\Rightarrow$ $b = 1$

So $a = 9$ $\Rightarrow$ $x > y > z$

16. Obvious

17. (A) $\because$ equal numbers are not always in A.P., G.P. and in H.P.
for example 0, 0, 0, ...

(B) $\frac{a-b}{b-c} = \frac{a}{c}$ $\Rightarrow ac - bc = ba - a\,c$ $\Rightarrow$ $2ac - bc = ab$ $\Rightarrow$ $b = \frac{2\ ac}{a+c}$

consider

$a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ in A.P. $\Rightarrow$ $b - a = c - b$ $\Rightarrow$ $2b = a + c$

So statemet is false.

(C) Let numbers are a, b

a, G_1, G_2, b or $A = \frac{a+b}{2}$

$r = \left(\frac{b}{a}\right)^{\frac{1}{3}}$; $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{3}}$, $G_2 = a.\left(\frac{b}{a}\right)^{\frac{2}{3}}$

$\therefore$ $\frac{G_1^3 + G_2^3}{G_1G_2} = \frac{a^3.\frac{b}{a} + a^3.\frac{b^2}{a^2}}{a^2.\frac{b}{a}} = \frac{a^2b + ab^2}{ab} = a + b = 2A$

(D) Let $T_{k+1} = ar^k$ and $T'_{k+1} = br^k$. Since $T''_{k+1} = ar^k + br^k = (a + b)\,r^k$,

$\therefore$ T''_{k+1} is general term of a G.P.

18. $\frac{\frac{a+b}{2}}{\sqrt{ab}} = \frac{2}{1}$ $\Rightarrow$ $\frac{a+b}{2\sqrt{ab}} = \frac{2}{1}$ use compendendo and dividendo rule

$\Rightarrow$ $\frac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}} = \frac{3}{1}$ $\Rightarrow$ $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{3}}{1}$ $\Rightarrow$ $\frac{2\sqrt{a}}{2\sqrt{b}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\Rightarrow$ $\frac{a}{b} = \frac{3+1+2\sqrt{3}}{3+1-2\sqrt{3}}$ $\Rightarrow$ $\frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}} = \frac{(2+\sqrt{3})\ (2+\sqrt{3})}{4-3} = 7 + 4\sqrt{3}$ **Ans**

19. $\sum_{r=1}^{n} r\,(r + 1)\,(2r + 3) = an^4 + bn^3 + cn^2 + dn + e$

$\sum_{r=1}^{n} (r^2+r)\,(2r + 3) = \sum_{r=1}^{n} (2r^3 + 5r^2 + 3r) = 2.\,\frac{n^2(n+1)^2}{4} + 5\,\frac{n(n+1)\ (2n+1)}{6} + 3\,\frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{5}{3}(2n+1)+3\right] = \frac{n(n+1)}{2}\left[\frac{6(n^2+n)+10(2n+1)+18}{6}\right] = \frac{n(n+1)}{12}$

$[6n^2 + 26n + 28] = \frac{1}{12}[6n^4 + 26n^3 + 28n^2 + 6n^3 + 26n^2 + 28n] = \frac{1}{12}[6n^4 + 32n^3 + 54n^2 + 28n]$

$a = \frac{6}{12}$; $b = \frac{32}{12}$; $c = \frac{54}{12}$; $d = \frac{28}{12} = \frac{7}{3}$; $e = 0$ so $a + c = b + d$

$b - \frac{2}{3} = \frac{32}{12} - \frac{2}{3} = \frac{24}{12}$; $c - 1 = \frac{42}{12}$

so a, b–2/3, c–1 are in A.P & $\frac{c}{a} = \frac{54}{6} = 9$ is an integer

20. Roots are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. ; A.M. = G.M. = 2.

Hence, all the roots are equal.

21. $6a = (a_3 - a_2) - (a_2 - a_1)$

$\Rightarrow \quad a = \frac{a_1 + a_3 - 2a_2}{6}$

PART - IV

1. $g(n) - f(n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}\left(\frac{2n+1}{3} - 1\right) = \frac{n(n+1)}{2}\frac{2n-2}{3}$

$= \frac{n(n+1)(n-1)}{3} = \frac{(n-1)\ n\ (n+1)}{3}$

for n = 2 $\frac{(n-1)\ n\ (n+1)}{3} = \frac{1.2.3}{3}$ which is divisible by 2 but not by 2^2

$\therefore$ greatest even integer which divides $\frac{(n-1)\ n\ (n+1)}{3}$,

for every $n \in N, n \geq 2$, is 2

2. $f(n) + 3g(n) + h(n) = \frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{2} + \left(\frac{n(n+1)}{2}\right)^2$

$= \frac{n(n+1)}{2}\left(1 + 2n + 1 + \frac{n(n+1)}{2}\right) = (1 + 2 + 3 + \ldots\ldots + n)\left(2n + 2 + \frac{n(n+1)}{2}\right)$

$\Rightarrow$ for all $n \in N$

Sol. **(3, 4)**

Let 1st term be a. and common difference is 2;

$T_{2n+1} = a + 4n = A$ (say) $r = \frac{1}{2}$

Middle term of AP = T_{n+1};

Middle term of GP = T_{3n+1}

$T_{n+1} = a + 2n \quad \Rightarrow \quad T_{3n+1} = A.\ r^n = \frac{(a+4n)}{2^n}\ (a + 2n) = \frac{a+4n}{2^n}$

$$\Rightarrow \quad 2^n a + 2n2^n = a + 4n$$

$$a = \frac{4n - 2n \ . \ 2^n}{2^n - 1} \Rightarrow T_{2n+1} = a + 4n = \frac{4n - 2n \ . \ 2^n}{2^n - 1} + 4n = \frac{2n \ . \ 2^n}{2^n - 1} = \frac{2^{n+1}n}{2^n - 1}$$

$$T_{3n+1} = \frac{a + 4n}{2^n} = \frac{2^{n+1}n}{2^n(2^n - 1)} = \frac{2n}{2^n - 1}$$

Sol. **(5 to 7)**

Let first term is 'a'

(5) $a(1-r)^2 = 36$

$\Rightarrow$ r can be 2, 3, 4, 7, –1, –2, –5

(6) $\frac{a}{1-r} = \frac{7}{3} \Rightarrow r = \frac{7-3a}{7}$

$\Rightarrow$ a can be 1, 2, 3, 4 and p can be 4, 1, –2, –5

(7) $ar^{n-1}(1-r)^6 = ar^{n-1}(1-r)^2$

$\Rightarrow (1-r)^4 = 1 \quad \Rightarrow r = 2$

EXERCISE # 3

PART - I

1. **(C)**

$S_n = cn^2$; $S_{n-1} = c(n-1)^2 = cn^2 + c - 2cn$

$T_n = 2cn - c$; $T_n^2 = (2cn - c)^2 = 4c^2n^2 + c^2 - 4c^2n$

$$\text{Sum} = \sum T_n^2 = \frac{4c^2 \ . \ n(n+1)(2n+1)}{6} + nc^2 - 2c^2n(n+1)$$

$$= \frac{2c^2n(n+1)(2n+1) + 3nc^2 - 6c^2n(n+1)}{3} = \frac{nc^2[4n^2 + 6n + 2 + 3 - 6n - 6]}{3} = \frac{nc^2(4n^2 - 1)}{3}$$

2. $\sum_{k=2}^{100} \left|(k^2 - 3k + 1) \ S_k\right|$

for k = 2 $\quad |(k^2 - 3k + 1) S_k| = 1$

$$\sum_{k=3}^{100} \left|\frac{k-1}{(k-2)\ !} - \frac{k-1+1}{(k-1)\ !}\right|$$

$$\sum_{k=3}^{100} \frac{1}{(k-3)\ !} + \frac{1}{(k-2)\ !} - \frac{1}{(k-2)\ !} - \frac{1}{(k-1)\ !}$$

$$\sum_{k=3}^{100} \left(\frac{1}{(k-3)\ !} - \frac{1}{(k-1)\ !}\right)$$

$$S = 1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{1\ !} - \frac{1}{3!}\right) + \left(\frac{1}{2!} - \frac{1}{4!}\right) + \left(\frac{1}{3!} - \frac{1}{5!}\right) + \left(\frac{1}{4!} - \frac{1}{6!}\right) + ... \left(\frac{1}{94!} - \frac{1}{96!}\right)$$

$$+ \left(\frac{1}{95!} - \frac{1}{97!}\right) + \left(\frac{1}{96!} - \frac{1}{98!}\right) + \left(\frac{1}{97!} - \frac{1}{99\ !}\right) = 2 - \frac{1}{98!} - \frac{1}{99!}$$

$$\therefore \quad E = \frac{100^2}{100!} + 3 - \frac{1}{98!} - \frac{1}{99.98!} = \frac{100^2}{100!} + 3 - \frac{100}{99!} = \frac{100^2}{100.99!} + 3 - \frac{100}{99!} = 3$$

3. $a_1 = 15$

$\frac{a_k + a_{k-2}}{2} = a_{k-1}$ for k = 3, 4,....,11 $\Rightarrow$ $\quad a_1, a_2, ..., a_{11}$ are in AP $a_1 = a = 15$

$\frac{a_1^2 + a_2^2 + ... + a_n^2}{11} = 90 \quad \Rightarrow \quad \frac{(15)^2 + (15+d)^2 + ... + (15+10d)^2}{11} = 90$

$\Rightarrow$ $9d^2 + 30d + 27 = 0$ $\Rightarrow$ $d = -3$ or $-\frac{9}{7}$

Since $27 - 2a_2 > 0 \Rightarrow a_2 < \frac{27}{2} \Rightarrow$ $d = -3$ $\frac{a_1 + a_2 + ... + a_{11}}{11} = \frac{11}{2}\frac{[30 + 10(-3)]}{11} = 0$

4. $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}[6 + (5n-1)d]}{\frac{n}{2}[6 + (n-1)d]} = \frac{5[(6-d) + 5nd]}{[(6-d) + nd]}$; $d = 6$ or $d = 0$

Now if $d = 0$ then $a_2 = 3$ else $a_2 = 9$ for single choice more appropriate choice is 9, but in principal, question seems to have an error.

$\therefore$ $a_2 = 3 + 6 = 9$

5. A.M. $\geq$ G.M.

$$\frac{\frac{1}{a^5} + \frac{1}{a^4} + \frac{1}{a^3} + \frac{1}{a^3} + \frac{1}{a^3} + 1 + a^8 + a^{10}}{8} \geq \left(\frac{1}{a^5}.\frac{1}{a^4}.\frac{1}{a^3}.\frac{1}{a^3}.\frac{1}{a^3}.1.a^8.a^{10}\right)^{1/8}$$

$\Rightarrow$ $\frac{1}{a^5} + \frac{1}{a^4} + \frac{3}{a^3} + 1 + a^8 + a^{10} \geq 8(1)^{1/8}$

$\Rightarrow$ minimum value of $\frac{1}{a^5} + \frac{1}{a^4} + \frac{3}{a^3} + 1 + a^3 + a^{10} = 8$, at $a = 1$

6. Corresponding A.P. $\frac{1}{5}$,..................... $\frac{1}{25}$ (20^{th} term)

$\frac{1}{25} = \frac{1}{5} + 19d$ $\Rightarrow$ $d = \frac{1}{19}\left(\frac{-4}{25}\right) = -\frac{4}{19 \times 25}$ $a_n < 0$

$\frac{1}{5} - \frac{4}{19 \times 25} \times (n-1) < 0$ $\Rightarrow$ $\frac{19 \times 5}{4} < n - 1$ $\Rightarrow n > 24.75$

7.* $S_n = \sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2 = -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 + ...$

$= (3^2 - 1) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2)......$ $= 2\,[\underbrace{4 + 6 + 12 + 14 + 20 + 22 + ...}_{2n\ \text{terms}}]$

$= 2[(4 + 12 + 20 ......) + (6 + 14 + 22 ....)]$

n terms n terms

$= 2\left[\frac{n}{2}(4 \times 2 + (n-1)8)\frac{n}{2}(2 \times 6 + (n+1)8)\right] = 2[n(4 + 4n - 4) + n(6 + 4n - 4)]$

$= 2(4n^2 + (4n + 2)n) = 2(8n^2 + 2n) = 4n(4n + 1)$

(A) 1056 = 32 × 33 n = 8

(B) 1088 = 32 × 34

(C) 1120 = 32 × 35

(D) 1332 = 36 × 37 n = 9

8. Numbers removed are k and k + 1. now $\frac{n(n+1)}{2} - k - (k + 1) = 1224$

$\Rightarrow n^2 + n - 4k = 2450$ $\Rightarrow n^2 + n - 2450 = 4k \Rightarrow (n + 50)(n - 49) = 4k$ $\Rightarrow n > 49$

Alternative

$\therefore$ To satisfy this equation n should be of the form of (4p + 1) or (4p + 2) taking n = 50

$\Rightarrow$ 4k = 100 $\Rightarrow$ k = 25

$\therefore$ k – 20 = 5

Now if we take n = 53 $\Rightarrow$ k = 103 $\Rightarrow$ n < k
so not possible. Hence n $\geq$ 53 will not be possible.

9. Let b = ar, c = ar² $\Rightarrow$ r is Integers. Also $\frac{a+ar+ar^2}{3}$ = ar + 2 $\Rightarrow$ a + ar² = 2ar + 6 $\Rightarrow$ a(r – 1)² = 6

$\Rightarrow$ r must be 2 and a = 6. Thus $\frac{a^2+a-14}{a+1} = \frac{36+6-14}{7}$ = 4 Ans.

10. $\frac{S_7}{S_{11}} = \frac{6}{11}$ $\frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11}$ Given 130 < a + 6d < 140 $\frac{7(a+3d)}{11(a+5d)} = \frac{6}{11}$

7a + 21d = 6a + 30d $\Rightarrow$ 130 < 15d < 140 a = 9d Hence d = 9 a = 81

Alternative :

Let the AP be a, a + d, a + 2d, ...where a , d $\in$ N Given $\frac{S_7}{S_{11}} = \frac{6}{11}$ and 130 < a + 6d < 140..... (2)

$\Rightarrow \frac{\frac{7}{2}\{2a+6d\}}{\frac{11}{2}\{2a+10d\}} = \frac{6}{11} \Rightarrow \frac{14a+42d}{22a+110d} = \frac{6}{11}$ $\Rightarrow$ 154a + 462d = 132a + 660d $\Rightarrow$ 22a = 198d $\Rightarrow$ a = $\frac{99d}{11}$ = 9d

(2) $\Rightarrow$ $\therefore$ 130 < 9d + 6d < 140 $\Rightarrow$ 8.6 < d < 9.3
$\therefore$ d = 9

11. $4\alpha x^2 + \frac{1}{x} \geq 1 \Rightarrow y = 4\alpha x^2 + \frac{1}{x} \Rightarrow y' = \frac{dy}{dx} = 8\alpha x - \frac{1}{x^2} = 0 \Rightarrow x = \left(\frac{1}{8\alpha}\right)^{1/3}$

$\Rightarrow f(x) = \frac{4\alpha x^3+1}{x} = \frac{1/2+1}{1/8(8\alpha)^{1/3}} \Rightarrow \frac{3}{2}\left(\frac{1}{8\alpha}\right)^{1/3} \Rightarrow \alpha^{1/3} \geq 1/3 \Rightarrow \alpha \geq \frac{1}{27}$

12. $\log_e b_1, \log_e b_2, \log_e b_3, \ldots\ldots \log_e b_{101}$ are in A.P. $b_1, b_2, b_3, \ldots\ldots, b_{101}$ are in G.P.

Given : $\log_e(b_2) - \log_e(b_1) = \log_e(2) \Rightarrow \frac{b_2}{b_1}$ = 2 = r (common ratio of G.P. $a_1, a_2, a_3, \ldots\ldots a_{101}$ are in A.P.

$a_1 = b_1$ = a $b_1 + b_2 + b_3 + \ldots\ldots b_{51}$ = t, S = $a_1 + a_2 + \ldots\ldots + a_{51}$

t = sum of 51 terms of G.P. = $b_1 = b_1\frac{(r^{51}-1)}{r-1} = \frac{a(2^{51}-1)}{2-1}$ a(2⁵¹ –1)

s = sum of 51 terms of A.P. = $\frac{51}{2}$ [2a₁ + (n–1)d] = $\frac{51}{2}$ (2a + 50d)

Given $a_{51} = b_{51}$ a + 50d = a(2)⁵⁰ 50d = a(2⁵⁰ – 1)

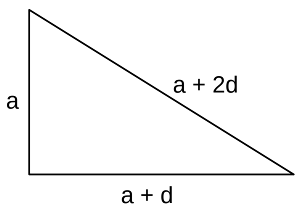
Hence s = $a\frac{51}{2}[2^{50}+1] \Rightarrow s = a\left(51.2^{49} + \frac{51}{2}\right)$

$s = 2\left(4.2^{49} + 47.2^{49} + \frac{51}{2}\right) \Rightarrow s = a\left((2^{51}-1) + 47.2^{49} + \frac{53}{2}\right)$ $s - t = a\left(47.2^{49} + \frac{53}{2}\right)$

Clearly s > t

$a_{101} = a_1 + 100d = a + 2a.2^{50} - 2a = a(2^{51} - 1)$ $b_{101} = b_1 r^{100} = a.2^{100}$ Hence $b_{101} > a_{101}$

13.



$\frac{1}{2}$a(a + d) = 24 $\Rightarrow$ a(a + d) = 48 ....(1)

a² + (a + d)² = (a + 2d)² $\Rightarrow$ 3d² + 2ad – a² = 0 (3d – a)(a + d) = 0

$\Rightarrow$ 3d = a ($\because$ a + d $\neq$ 0) $\Rightarrow$ d = 2 a = 6 so smallest side = 6

14. P = {1, 6, 11, ...}

Q = {9, 16, 23,.........}

Common terms: 16, 51, 86

$t_p = 16 + (p - 1)35 = 35p - 19 \leq 10086 \Rightarrow p \leq 288.7$

$\therefore$ n (P $\cup$ Q) = n(P) + n (Q) – n (P $\cap$ Q) = 2018 + 2018 – 288 = 3748

15. First series is {1, 4, 7, 10, 13,……..}

Second series is {2, 7, 12, 17, …….}

Third series is {3, 10, 17, 24, ………}

See the least number in the third series which leaves remainder 1 on dividing by 3 and leaves remainder 2 on dividing by 5.

$\Rightarrow$ 52 is the least number of third series which leaves remainder 1 on dividing by 3 and leaves remainder 2 on dividing by 5

Now, A = 52

D is L.C. M. of (3, 5, 7) = 105 $\Rightarrow$ A + D = 52 + 105 = 157

PART - II

1. $a_1 + a_2 + ... + a_N = 4500$ notes

$a_1 + a_2 + ... + a_{10} = 150 \times 10 = 1500$ notes = 4500 – 1500 = 3000 notes

$a_{11} + a_{12} + ... + a_n = 3000 \Rightarrow 148 + 146 ...... = 3000$

$[2 \times 148 + (n - 10 - 1)(-2)] = 3000 \Rightarrow n = 34, 135$

$a_{34} = 148 + (34 - 1)(-2) = 148 - 66 = 82$

$a_{135} = 148 + (135 - 1)(-2) = 148 - 268 = -120 < 0$. so answer in 34 minutes is taken

Hence correct option is (1)

2. a = Rs. 200 ; d = Rs. 40 $\Rightarrow$ savings in first two months = Rs. 400

remained savings = 200 + 240 + 280 + ... upto n terms

$= \frac{n}{2}[400 + (n-1)\ 40] = 11040 - 400 \Rightarrow 200n + 20n^2 - 20n = 10640$

$20n^2 + 180n - 10640 = 0 \Rightarrow n^2 + 9n - 532 = 0$

$(n + 28)(n - 19) = 0 \Rightarrow n = 19$

$\therefore$ no. of months = 19 + 2 = 21.

3. Let A.P. be a, a + d, a + 2d, ...

$a_2 + a_4 + ... + a_{200} = \alpha \Rightarrow \frac{100}{2}[2(a + d) + (100 - 1)d] = \alpha$ .... (i)

and $a_1 + a_3 + a_5 + ... + a_{199} = \beta \Rightarrow \frac{100}{2}[2a + (100 - 1)d] = \beta$ .....(ii)

on solving (i) and (ii) $d = \frac{\alpha - \beta}{100}$

4. $\frac{7}{10} + \frac{77}{100} + \frac{777}{10^3} + ...+$ up to 20 terms $= 7\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{10^3} + \text{.....up to 20 terms}\right]$

$= \frac{7}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \text{.....up to 20 terms}\right] =$

$\frac{7}{9}\left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \text{.....up to 20 terms}\right]$

$$= \frac{7}{9}\left[20 - \frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^{20}\right)}{1-\frac{1}{10}}\right] = \frac{7}{9}\left[20 - \frac{1}{9}\left(1-\left(\frac{1}{10}\right)^{20}\right)\right] = \frac{7}{9}\left[\frac{179}{9}+\frac{1}{9}\left(\frac{1}{10}\right)^{20}\right] = \frac{7}{81}\left[179+(10)^{-20}\right]$$

5. Let $s = (10)^9 + 2(11)^1 (10)^8 + 3 (11)^2(10)^7 + \ldots\ldots + 10(11)^9$

$\therefore s\,\frac{11}{10} = (11)^1 (10)^8 + 2(11)^2 (10)^7 + \ldots\ldots + 9(11)^9 + (11)^{10}$

substract $\left(1-\frac{11}{10}\right) s = (10)^9 + (11)^1 (10)^8 + (11)^2 (10)^7 + \ldots\ldots + (11)^9 - (11)^{10}$

$$\Rightarrow -\frac{1}{10} s = \frac{10^9\left\{1-\left(\frac{11}{10}\right)^{10}\right\}}{1-\frac{11}{10}} - (11)^{10} - \frac{1}{10} s = 10^9 \frac{\{10^{10}-11^{10}\}}{10^{10}} \times \frac{10}{-1} - (11)^{10}$$

$\Rightarrow -\frac{1}{10} s = -10^{10} + 11^{10} - 11^{10}$ $\quad\therefore\quad s = 10^{11}$

$\therefore$ given $10^{11} = k(10)^9$ $\quad\therefore\quad k = 100$

6. a a_1 ar^2 G. P. ; a 2ar ar^2 A. P.

$L_1 ar = a + ar^2$; $4r = 1 + r^2$

$r^2 - 4r + 1 = 0$; $r = \frac{4 \mp 2\sqrt{3}}{2} = 2 + \sqrt{3}$, $2 - \sqrt{3}$

But r > 1 $\quad r = 2+\sqrt{3}$

7. $M = \frac{\ell + n}{2}$

ℓ, G_1, G_2, G_3, n are in G.P.

$r = \left(\frac{n}{\ell}\right)^{\frac{1}{4}}$

$G_1 = \ell \left(\frac{n}{\ell}\right)^{\frac{1}{4}}$ $G_2 = \ell\left(\frac{n}{\ell}\right)^{\frac{1}{2}}$ $\quad G_3 = \ell\left(\frac{n}{\ell}\right)^{\frac{3}{4}}$

$G_1^4 + 2G_2^4 + G_3^4$

$= \ell^4 \times \frac{n}{\ell} + 2\ell^4 \frac{n^2}{\ell^2} + \ell^4 \times \frac{n^3}{\ell^3}$

$= \ell^3 n + 2\ell^2 n^2 + \ell n^3$

$= n\ell(\ell^2 + 2n\ell + n^2)$

$= n\ell(\ell + n)^2$

$= 4m^2 n\ell$

8. $T_n = \frac{\frac{n^2(n+1)^2}{4}}{n^2}$

$T_n = \frac{1}{4}(n + 1)^2$

$T_n = \frac{1}{4}[n^2 + 2n + 1]$

$S_n = \sum_{n=1}^{n} T_n$

$S_n = \frac{1}{4}\left[\frac{n(n+1)\ (2n+1)}{6} + n(n+1) + n\right]$

n = 9

$S_9 = \frac{1}{4}\left[\frac{9\times10\times19}{6} + 9\times10+9\right] = \frac{1}{4}[285 + 90 + 9] = \frac{384}{4} = 96.$

9. a + d, a + 4d, a + 8d → G.P

∴ $(a + 4d)^2 = a^2 + 9ad + 8d^2$

⇒ $8d^2 = ad$ ⇒ a = 8d

∴ 9d, 12d, 16d → G.P. common ratio $r = \frac{12}{9} = \frac{4}{3}$

10. $\left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \left(\frac{24}{5}\right)^2 + ... \frac{8^2}{5^2} + \frac{12^2}{5^2} + \frac{16^2}{5^2} + \frac{20^2}{5^2} + \frac{24^2}{5^2} + ...$

$T_n = \frac{(4n+4)^2}{5^2}$ $S_n = \frac{1}{5^2}\sum_{n=1}^{10} 16(n+1)^2 = \frac{16}{25}\sum_{n=1}^{10}(n^2+2n+1)$

$= \frac{16}{25}\left[\frac{10\times11\times21}{6} + \frac{2\times10\times11}{2} + 10\right] = \frac{16}{25}\times505 = \frac{16}{5}$ m ⇒ m = 101

11. $225a^2 + 9b^2 + 25c^2 - 75ac - 45ab - 15bc = 0$

$(15a)^2 + (3b)^2 + (5c)^2 - (15a)(3b) - (3b)(5c) - (15a)(5c) = 0$

$\frac{1}{2}[(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2] = 0$

15a = 3b, 3b = 5c, 5c = 15a

5a = b, 3b = 5c, c = 3a

$\frac{a}{1} = \frac{b}{5} = \frac{c}{3} = \lambda$

a = λ, b = 5λ, c = 3λ

a, c, b are in AP

b, c, a are in AP

12. $f(x) = ax^2 + bx + c$

f(x + y) = f(x) + f(y) + xy

$a(x + y)^2 + b(x+ y) + c = ax^2 + bx + c + ay^2 + by + c + xy$

2axy = c + xy ∀x, y∈R

(2a – 1)xy – c = 0 ∀ x, y∈R

⇒ c = 0, $a = \frac{1}{2}$ a + b + c = 3

$\frac{1}{2} + b + 0 = 3$ $b = \frac{5}{2}$

∴ $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x$ $\sum_{n=1}^{10} f(n) = \frac{1}{2}\sum_{n=1}^{10} n^2 + \frac{5}{2}\sum_{n=1}^{10} n \Rightarrow \frac{1}{2}\times\frac{10\times11\times21}{6} + \frac{5}{2}\times\frac{10\times11}{2} = 330$

13. The given quadratic equation is

$nx^2 + x(1 + 3 + 5 + ... + (2n - 1)) + (1.\,2 + 2.3 + ... + (n - 1).n) - 10n = 0$

$\Rightarrow nx^2 + x(n^2) + \frac{n(n^2-1)}{3} - 10n = 0$ $\Rightarrow x^2 + x(n) + \frac{(n^2-1)}{3} - 10 = 0$ (roots α, β)

$(\alpha - \beta)^2 = 1$ $\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$ $\Rightarrow n^2 - 4\left(\frac{n^2-1}{3} - 10\right) = 1 \Rightarrow n = 11$

14. $a_1 + a_5 + a_9 + a_{13} + ... + a_{49} = 416$

$a_1 + (a_1 + 4d) + (a_1 + 8d) + (a_1 + 12d) + ... + (a_1 + 48d) = 416$

$13 a_1 + 4d(1+2+3+......+12) = 416$

$13\,a_1 + \frac{4d \times 12 \times 13}{2} = 416$

$13a_1 + 24 \times 13d = 416$

$a_1 + 24d = 32 \quad a_9 + a_{43} = 66 \quad a_1 + 8d + a_1 + 42d = 66$

$2a_1 + 50d = 66 \quad a_1 + 25d = 33 \quad d = 1 \quad a_1 = 8$

$a_1^2 + (a_1 + d)^2 + (a_1 + 2d)^2 + \ldots\ldots + (a_1 + 16d)^2 = 140\text{ m}$

$17a_1^2 + d^2(1^2 + 2^2 + \ldots + 16^2) + 2a_1d\,(1+2+3+\ldots\ldots+16) = 140m$

$17 \times 64 + \frac{16 \times 17 \times 33}{6} + \frac{2 \times 8 \times 1 \times 16 \times 17}{2} = 140\text{ m}$

$17 \times 64 + 8 \times 11 \times 17 + 8 \times 11 \times 17 = 140\text{ m}$

$17 \times 16 + 22 \times 17 + 2 \times 16 \times 17 = 35\text{ m}$

$272 + 374 + 544 = 35\text{ m}$

$1190 = 35\text{ m} \quad \Rightarrow \quad m = 34$

15. $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$

$A = 1^2 + 2.2^2 + \ldots + 2.20^2$

$= (1^2 + 2^2 + \ldots + 20^2) + (2^2 + 4^2 + \ldots + 20^2)$

$= \frac{20.21.41}{6} + 4.\frac{10.11.21}{6} = \frac{20.21}{6}\{41 + 22\} = 70 \times 63 = 4410$

$B = 1^2 + 2.2^2 + \ldots + 2.40^2$

$= (1^2 + 2^2 + \ldots + 40^2) + (2^2 + 4^2 + \ldots + 40^2)$

$= \frac{40.41.81}{6} + \frac{4.20.21.41}{6} = \frac{40.41}{6}(81 + 42) = \frac{40.41}{6} \times 123$

$= 20\,(41)^2 = 33620$

$B - 2A = 100\lambda \Rightarrow \lambda = \frac{33620 - 8820}{100} = \frac{24800}{100} = 248$

16. This series can be written as

$\frac{3(1^2)}{3} + \frac{6(1^2 + 2^2)}{5} + \frac{9(1^2 + 2^2 + 3^2)}{7} + \frac{12(1^2 + 2^2 + 3^2 + 4^2)}{9} + \ldots$

$T_r = \frac{3r}{(2r+1)}(1^2 + 2^2 + 3^2 + \ldots r^2)$

$T_r = \frac{3r}{(2r+1)} \frac{r(r+1)(2r+1)}{6}$

$T_r = \frac{1}{2}r^2(r + 1)$

sum of n term $\sum_{i=1} T_r = \frac{1}{2}(\Sigma r^3 + \Sigma r^2) = \frac{1}{2}\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right]$

hence sum of 15 term $= \frac{1}{2}\left[\left(\frac{15\;.\;16}{2}\right)^2 + \frac{15.16.31}{6}\right] = \frac{1}{2}[14400 + 1240] = 7820$

17. $5, 5r, 5r^2$ sides of triangle,

$5 + 5r > 5r^2 \quad \ldots (1)$

$5 + 5r^2 > 5r \quad \ldots (2)$

$5r + 5r^2 > 5 \quad \ldots (3)$

from (1) $r^2 - r - 1 < 0$, (1)

$\left[r - \left(\frac{1+\sqrt{5}}{2}\right)\right]\left[r - \left(\frac{1-\sqrt{5}}{2}\right)\right] < 0 \; r \in \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right) \quad \ldots(4)$

from (2) , (2)

$r^2 - r + 1 > 0 \Rightarrow r \in R$ ....(5)

from (3), (3) $r^2 + r - 1 > 0$ so, $\left(r + \frac{1+\sqrt{5}}{2}\right)\left(r + \frac{1-\sqrt{5}}{2}\right) > 0$ $r \in \left(-\infty, -\frac{1+\sqrt{5}}{2}\right) \cup \left(-\frac{1-\sqrt{5}}{2}, \infty\right)$ ...(6)

from (4), (5), (6), $r \in \left(\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ now check options

18. Natural numbers between 100 & 200.

101, 102, ..., 199.

Either divide by 7 or divide by 13.

(sum of numbers (divide by 7) + (sum of number divide by 13) – (sum of number of divide by 91)

$$\sum_{r=1}^{14}(98+7r) + \sum_{r=1}^{8}(91+13r) - (182) = \left(98\times14 + 7.\frac{14\times15}{2}\right) + \left((91\times8) + 13\times\frac{8\times9}{2}\right) - 182$$

= 1372 + 735 + 728 + 468 – 182 = 3121

19. $a_1, a_2, a_3, \ldots a_{50}$ in A.P.

$a_6 = 2$ $\therefore$ $a_1 + 5d = 2$

$a_1 a_4 a_5 = a_1(a_1 + 3d)(a_1 + 4d)$

$= a_1(2 - 2d)(2 - d)$

$= -2((5d - 2)(d - 1)(d - 2))$

$= -2(5d^3 - 17d^2 + 16d - 4)$

$\frac{dA}{d(d)} = -2(15d^2 - 34d + 16)$

$= -2(5d - 8)(3d - 2)$

– + –

2/3 8/5

Maximum occurs at $d = \frac{8}{5}$

20. Let $b = ar$, $c = ar^2$

Hence $3a + 15ar^2 = 14ar$

$15r^2 - 14r + 3 = 0$

$15r^2 - 9r - 5r + 3 = 0$

$(3r - 1)(5r - 3) = 0$

$r = \frac{1}{3}, \frac{3}{5}$ $\Rightarrow$ $r = \frac{1}{3}$

AP is $3a, \frac{7}{3}a, \frac{5}{3}a, a, \ldots\ldots$

$\Rightarrow$ 4^{th} terms is a

21. $\frac{(49)^{126} - 1}{48} = \frac{((49)^{63} + 1)(49^{63} - 1)}{48}$

22. $f(x) = \left(\frac{2^{1-x} + 2^{1+x} + 3^x + 3^{-x}}{2}\right)$

Using AM $\geq$ GM

$f(x) \geq 3$

23. Let GP is $a, ar, ar^2 \ldots\ldots$

$\sum_{n=1}^{100} a_{2n+1} = a_3 + a_5 + \ldots a_{201} = 200 \Rightarrow \frac{ar^2(r^{200} - 1)}{r^2 - 1} = 200$ ....(1)

$\sum_{n=1}^{100} a_{2n}$ = $a_2 + a_4 + \ldots\ldots a_{200} = 100 = \frac{ar(r^{200}-1)}{r^2-1} = 100$ ....(2)

Form (1) and (2) r = 2

add both

$\Rightarrow a_2 + a_3 + \ldots\ldots\ldots a_{200} + a_{201} = 300$ $\Rightarrow$ $r(a_1 + \ldots\ldots\ldots\ldots a_{200}) = 300$

$\sum_{n=1}^{200} a_n = \frac{300}{r} = 150$

24. $y = 1 + \cos^2\theta + \cos^4\theta + \ldots\ldots$

$\Rightarrow y = \frac{1}{1-\cos^2\theta} \Rightarrow \frac{1}{y} = \sin^2\theta$

$x = \frac{1}{1-(-\tan^2\theta)} = \frac{1}{\sec^2\theta}$ $\Rightarrow \cos^2\theta = x$ $\Rightarrow \frac{1}{y} + x = 1$ $\Rightarrow y(1-x) = 1$

HIGH LEVEL PROBLEMS (HLP)

PART - I

1. $T_m \equiv a + (m-1)d = \sqrt{2}$ ....(1)

$T_n \equiv a + (n-1)d = \sqrt{3}$ ....(2)

$T_p \equiv a + (p-1)d = \sqrt{5}$ ....(3)

$\frac{(2)-(1)}{(3)-(2)} \Rightarrow \frac{n-m}{p-n} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$. Not possible.

2. Let a denotes first term and d common difference of the A.P. then a_k, the k^{th} term of the A.P.

$a_k = a + (k-1)d$; $a_1 + a_2 + a_3 + \ldots + a_m = a_{m+1} + a_{m+2} + \ldots + a_{m+n}$

$2S_m = S_{m+n}$; $2.\ \frac{m}{2}[2a + (m-1)d] = \frac{m+n}{2}[2a + (m+n-1)d]$

$\frac{2m}{m+n} = \frac{2a+(m+n-1)\ \ d}{2a+(m-1)\ \ d} = 1 + \frac{nd}{2a+(m-1)\ \ d} \Rightarrow \frac{nd}{2a+(m-1)\ \ d} = \frac{m-n}{m+n}$ ......(i)

similarly $\frac{pd}{2a+(m-1)d} = \frac{m-p}{m+p}$ ......(ii) divide (i) by (ii)

$\frac{n}{p} = \frac{m-n}{m+n} \cdot \frac{m+p}{m-p} \Rightarrow \frac{(m+n)(m-p)}{p} = \frac{(m+p)(m-n)}{n} \Rightarrow \frac{(m+n)(m-p)}{mp} = \frac{(m+p)(m-n)}{mn}$

$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$ Hence Proved

3. $a = A + (p-1)d$ $\Rightarrow$ $d = \frac{a-b}{p-q}$ $b = A + (q-1)d$

$S = \frac{p+q}{2}[2A + (p+q-1)d] = \frac{p+q}{2}[A + (p-1)d + A + (q-1)d + d]$

$= \frac{p+q}{2}[a + b + d] = \frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$

4. Given $\frac{p}{2}[2a + (p-1)d] = 0.$ so $2a + (p-1)d = 0$ $\Rightarrow$ $d = -\frac{2a}{(p-1)}$

Now sum of next q terms are = sum of (p + q) terms of this A.P.

$= \frac{p+q}{2}[2a + (p+q-1)d] = \frac{p+q}{2}[2a + (p-1)d + qd] = \frac{p+q}{2}.\ q.\ d = -\frac{a\ \ (p+q)\ \ q}{p-1}$

5. $\frac{a+be^y}{a-be^y}+1 = \frac{b+c\ \ e^y}{b-c\ \ e^y}+1 = \frac{c+d\ \ e^y}{c-d\ \ e^y}+1;$ $\frac{2a}{a-b\ \ e^y} = \frac{2b}{a-c\ \ e^y} = \frac{2c}{c-d\ \ e^y}$

$1 - \frac{b}{a} e^y = 1 - \frac{c}{b} e^y = 1 - \frac{d}{c} e^y \quad \Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \quad \Rightarrow$ a, b, c, d are in G.P.

6. Let first term of AP = a and common difference = d

$\therefore \quad \frac{10}{2}[2a + 9d] = 155 \quad \Rightarrow \quad .2a + 9d = 31$

GP first term = a′ and common ratio = r a′ + a′r = 9 ⇒ a′(1 + r) = 9 given a = r and a′ = d

$\Rightarrow \quad 2r + 9a' = 31 \Rightarrow 2r + 9.\frac{9}{1+r} = 31 \Rightarrow \quad 2r^2 - 29r + 50 = 0 \Rightarrow r = 2, 25/2.$

(i) if r = 2, So a = 2 ⇒ a′ = 3 = d

(ii) if $r = \frac{25}{2} \Rightarrow a = \frac{25}{2} \Rightarrow a' = \frac{9}{\frac{27}{2}} = \frac{2}{3} = d$

So the series 3, 6, 12.........second $\frac{2}{3}, \frac{25}{3}, \frac{625}{6}$

7. (i) 1, (2, 3), (4, 5, 6, 7), (8, 9, ...,15) ....

Number of terms in n^{th} group = 2^{n-1}; 1^{st} term in n^{th} group = 2^{n-1}

So, Sum of terms in n^{th} group $\frac{2^{n-1}}{2} = [2.2^{n-1} + (2^{n-1} - 1)\,1] = 2^{n-2}\,[2^n + 2^{n-1} - 1]$

(ii) (1) , (2, 3, 4), (5, 6, 7, 8, 9) ; 1^{st} term in n^{th} group let T_n

1 2 5 10

 1 3 5

  2 2

$T_n = a + bn + cn^2 \Rightarrow T_1 = a + b + c = 1$ .... (i)

$T_2 = a + 2b + 4c = 2$ .... (ii)

and $T_3 = a + 3b + 9c = 5$ .... (iii)

a = 2, b = – 2, c = 1 On solving (i), (ii) and (iii), we get

So I^{st} term is $T_n = (2 - 2n + n^2)$. Number of terms in n^{th} group = (2n – 1)

$\therefore$ sum of terms in n^{th} group $\frac{2n-1}{2} = [2\,(2 - 2n + n^2) + 2n - 2]$

$= (2n - 1)\,[n^2 - n + 1] = 2n^3 - 2n^2 + 2n - n^2 + n - 1 = n^3 + (n - 1)^3$

8. $a, A_1, A_2, b \to$ A.P. $\Rightarrow A_1 + A_2 = a + b.$

$a, G_1, G_2, b \to$ G.P. $\Rightarrow G_1\,G_2 = ab$

$a, H_1, H_2, b \to$ H.P. $\Rightarrow \frac{ab}{a+b} = \frac{H_1 \;\; H_2}{H_1 + H_2}$

$\Rightarrow \quad \frac{A_1 + A_2}{H_1 + H_2} = \frac{G_1 \;\; G_2}{H_1 \;\; H_2} \Rightarrow \frac{1}{H_1} = \frac{1}{a} + d \quad \frac{1}{b} = \frac{1}{a} + 3d \quad \Rightarrow \quad \frac{a-b}{3\;a\;b} = d$

$\therefore \quad \frac{1}{H_1} = \frac{a+2b}{3ab} \quad \frac{1}{H_2} = \frac{2a+b}{3\;ab} \quad \frac{1}{H_1H_2} = \frac{(a+2b)\;\;(2a+b)}{9a^2b^2} \Rightarrow \frac{G_1G_2}{H_1\;\;H_2} = \frac{(a+2b)\;\;(2a+b)}{9\;\;ab}$

9. $T_k = k.2^{n+1-k}$ and $S_n = \left(\frac{n+1}{4}\right)\left(2^{n+1} - n - 2\right)$

Now, $S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1} \sum_{k=1}^{n} k \;.\; 2^{-k}$; $S_n = 2^{n+1}.\;2\;.\left[1 - \frac{1}{2^n} - \frac{n}{2^{n+1}}\right]$ (sum of A.G.P.)

$\left(\frac{n+1}{4}\right)\left(2^{n+1} - n - 2\right) = 2.\,[2^{n+1} - 2 - n] \Rightarrow \quad \frac{n+1}{4} = 2 \quad \Rightarrow \quad n = 7$

10. Let G_m be the geometric mean of $G_1, G_2, ...,G_n$ $\quad \Rightarrow \quad G_m = (G_1 \cdot G_2 .....G_n)^{1/n}$

$= [(a_1)(a_1a_1r)^{1/2}.(a_1.a_1r.a_1r^2)^{1/3}...\,(a_1.a_1r.a_1r^2\,...a_1r^{n-1})^{1/n}]^{1/n}$

where r is the common ratio of G.P. $a_1, a_2, ...,a_n. = [(a_1.a_1...n \text{ times})$;त१२. $r^{3/3}.\ r^{6/4}....\ r^{\frac{(n-1)n}{2n}})]^{1/n}$

$\Rightarrow \quad a_1 . \left[r^{\frac{1}{2}\left[\frac{(n-1)\,n}{2}\right]} \right]^{1/n} = a_1 \left[r^{\frac{n-1}{4}} \right]$. Now, $\quad A_n = \dfrac{a_1 + a_2 + ... + a_n}{n} = \dfrac{a_1(1-r^n)}{n\ (1-r)}$

and $\quad H_n = \dfrac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} ... + \dfrac{1}{a_n}\right)} = \dfrac{n}{\dfrac{1}{a_1}\left(1 + \dfrac{1}{r} + ... + \dfrac{1}{r^{n-1}}\right)} = \dfrac{a_1\ n(1-r)\ r^{n-1}}{1-r^n}$

Again $\quad A_n.H_n = \dfrac{a_1(1-r^n)}{n\ (1-r)} \times \dfrac{a_1 n\ (1-r)\ r^{n-1}}{(1-r^n)} = a_1^2\ r^{n-1} \quad \Rightarrow \prod_{k=1}^{n} A_k H_k = \prod_{k=1}^{n} (a_1^2\ r^{k-1})$

$= (a_1^2.\ a_1^2 ...\ n\ \text{ times}) \times (r^0.r^1.r^2 ...r^{n-1}) = a_1^{2n}.r^{1+2+....(n-1)} = a_1^{2n}\ r^{\frac{n(n-1)}{2}} = [a_1\ r^{\frac{n-1}{4}}]^{2n}$

$= [G_m]^{2n} \Rightarrow \quad G_m = \left[\prod_{k=1}^{n} A_K H_K\right]^{1/2n} \quad \Rightarrow \quad G_m = (A_1 A_2 ... A_n.H_1H_2 ....H_n)^{1/2n}$

11. $2b = a + c$ ..... (1)

$q = \dfrac{2\ pr}{p+r}$ ..... (2)

and $b^2 q^2 = acrp$ ..... (3)

So $\left(\dfrac{a+c}{2}\right)^2 \left(\dfrac{2\ pr}{p+r}\right)^2 = acrp \quad \Rightarrow \dfrac{(p+r)^2}{pr} = \dfrac{(a+c)^2}{ac}$

$\Rightarrow \quad \dfrac{p}{r} + \dfrac{r}{p} + 2 = \dfrac{a}{c} + \dfrac{c}{a} + 2 \quad \Rightarrow \quad \dfrac{p}{r} + \dfrac{r}{p} = \dfrac{a}{c} + \dfrac{c}{a}$ **Ans.**

12. $\alpha + \beta = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} \quad \Rightarrow \quad \alpha + \beta = \dfrac{\alpha^2 + \beta^2}{\alpha^2\, \beta^2} \Rightarrow \quad -\dfrac{b}{a} = \dfrac{b^2 - 2ac}{c^2}$

$\Rightarrow \quad -bc^2 = ab^2 - 2a^2c \quad \Rightarrow \quad ab^2 + bc^2 = 2a^2c \quad \Rightarrow \quad \dfrac{b}{c} + \dfrac{c}{a} = \dfrac{2a}{b}$

So $\dfrac{c}{a}, \dfrac{a}{b}, \dfrac{b}{c}$ are in A.P. $\quad \Rightarrow \quad \dfrac{a}{c}, \dfrac{b}{a}, \dfrac{c}{b}$ are in H.P.

13. $b = \dfrac{2ac}{a+c}$; $c^2 = bd$, $d = \dfrac{c+e}{2}$; $\quad ab + bc = 2ac$

$c = \dfrac{ab}{2a-b}$; $\quad d = \dfrac{1}{2}\left(\dfrac{ab}{2a-b} + e\right)$

from $c^2 = bd$ $\left(\dfrac{ab}{2a-b}\right)^2 = \dfrac{b}{2}\left(\dfrac{ab}{2a-b} + e\right) \quad \Rightarrow \quad \dfrac{a^2b^2}{(2a-b)^2} = \dfrac{b}{2}\left(\dfrac{ab + 2ae - be}{2a-b}\right)$

$\Rightarrow \quad \dfrac{2a^2b}{(2a-b)^2} - \dfrac{ab}{(2a-b)} = e \quad \Rightarrow \quad e(2a-b)^2 = 2a^2b - ab(2a-b) \quad \Rightarrow \quad e(2a-b)^2 = ab^2$

14. $x + y + z = 15$ ....(i)

a, x, y, z, b are in AP

Suppose d is common difference $d = \dfrac{b-a}{4}$

$\therefore \quad x = a + \dfrac{b-a}{4} = \dfrac{b+3a}{4}$, $y = \dfrac{2b+2a}{4}$ and $z = \dfrac{3b+a}{4}$

on substituting the values of X, Y and Z in (i), we get

$\Rightarrow \quad \dfrac{6a+6b}{4} = 15$

$a + b = 10$ ..... (ii)

$\because \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{3}$ ....(iii)

and a, x, y, z, b are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in A.P.

$\therefore \quad \frac{1}{x} = \frac{1}{a} + \frac{\left(\frac{1}{b} - \frac{1}{a}\right)}{4}, \quad \frac{1}{y} = \frac{1}{a} + \frac{2}{4}\left(\frac{1}{b} - \frac{1}{a}\right)$ and $\frac{1}{z} = \frac{1}{a} + \frac{3}{4}\left(\frac{1}{b} - \frac{1}{a}\right)$

on substituting the value of $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ in (iii), we get

$\frac{3}{a} + \frac{6}{4}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{5}{3} \Rightarrow \frac{3}{2a} + \frac{3}{2b} = \frac{5}{3} \Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{10}{9}$ ..... (iv)

By equations (ii) & (iv), we get

a = 9, b = 1 or a = 1, b = 9

15. Let n HM's between a and c are $H_1, H_2, H_3, \ldots, H_n$

$\therefore \quad \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \ldots\ldots, \frac{1}{H_n}, \frac{1}{c}$ are in A.P.

suppose 'd' is common difference of this A.P.

$\therefore \quad \frac{1}{c} = \frac{1}{a} + (n+1)\,d \quad \Rightarrow \quad d = \frac{1}{(n+1)}\left(\frac{1}{c} - \frac{1}{a}\right)$

so $\frac{1}{H_1} = \frac{1}{a} + \frac{1}{(n+1)}\left(\frac{1}{c} - \frac{1}{a}\right) = \frac{nc+a}{(n+1)\;ca} \Rightarrow H_1 = \frac{(n+1)\;ca}{nc+a}$

and $\frac{1}{H_n} = \frac{1}{a} + \frac{n}{(n+1)}\left(\frac{1}{c} - \frac{1}{a}\right) = \frac{c+na}{(n+1)\;ca} \Rightarrow H_n = \frac{(n+1)\;ca}{c+na}$

$\therefore \quad H_1 - H_n = ac(n+1)\left[\frac{1}{nc+a} - \frac{1}{c+na}\right] = ac(n+1)\left[\frac{(a-c)\;(n-1)}{(nc+a)\;(c+na)}\right]$

$H_1 - H_n = \frac{ac\;(n^2-1)\;(a-c)}{n^2ac + n(a^2+c^2) + ac}$ ....(i)

$\because$ n is a root of $x^2(1 - ac) - x(a^2 + c^2) - (1 + ac) = 0$ we have $n^2(1 - ac) - n(a^2 + c^2) - (1 + ac) = 0$

$\Rightarrow \quad n^2ac + n(a^2 + c^2) + ac = (n^2 - 1)$ ....(ii)

$\therefore$ from (i) and (ii), we get So $H_1 - H_n = ac(a - c)$ Ans.

16. (i) $b^2c^2 + c^2a^2 + a^2b^2 \ge abc\,(a + b + c) \Rightarrow (bc - ca)^2 + (ab - bc)^2 + (ca - ab)^2 \ge 0$

$\therefore \quad b^2c^2 + c^2a^2 + a^2b^2 \ge abc\,(a + b + c)$

If a, b, x, y are real number and x, y > 0, then $\frac{a^2}{x} + \frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}$ ....(i)

as on solving it we have $(ay - bx)^2 \ge 0$

Similarly, we can extend the inequality to three pairs of numbers

i.e. $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}$

Now we use this result to following questions

(ii) As $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{ab+ac} + \frac{b^2}{bc+ba} + \frac{c^2}{ca+cb} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)}$

Now, we have to prove that $\frac{(a+b+c)^2}{2(ab+bc+ac)} \ge \frac{3}{2}$

or $a^2 + b^2 + c^2 \ge ab + bc + ca \Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 \ge 0$

(iii) L.H.S. = $\frac{(\sqrt{2})^2}{a+b} + \frac{(\sqrt{2})^2}{b+c} + \frac{(\sqrt{2})^2}{c+a} \ge \frac{(3\sqrt{2})^2}{2(a+b+c)}$ L.H.S. = $\frac{(\sqrt{2})^2}{a+b} + \frac{(\sqrt{2})^2}{b+c} + \frac{(\sqrt{2})^2}{c+a} \ge \frac{9}{a+b+c}$

17. A.M. = G.M.

$\therefore$ $x_1 = x_2 = x_3 = x_4 = 2$.

18. Let $\sqrt{a_1} = b_1$; $\sqrt{a_2 - 1} = b_2$; $\sqrt{a_3 - 2} = b_3$; .... $\sqrt{a_n - (n-1)} = b_n$

$\therefore$ $b_1 + b_2 + ... + b_n = \frac{1}{2}$ $[b_1^2 + (b_2^2 + 1) + (b_3^2 + 2) + ... + (b_n^2 + (n-1))] - \frac{n(n-3)}{4}$

$\therefore$ $\Sigma b_1 = \frac{1}{2}$ $[(b_1^2 + b_2^2 + b_3^2 + ...+ b_n^2) + (1 + 2 + 3 + ...+ (n-1)] - \frac{n(n-3)}{4}$

$\Rightarrow$ $2\Sigma b_1 = \Sigma b_1^2 + \frac{n(n-1)}{2} - \frac{n(n-3)}{2}$ $\Rightarrow$ $2\Sigma b_1 = \Sigma b_1^2 + n$

$\therefore$ $\Sigma b_1^2 - 2\Sigma b_1 + \Sigma 1 = 0$ $\Rightarrow$ $\sum_{i=1}^{n}(b_1 - 1)^2 = 0$ $b_1 - 1 = 0$ $\Rightarrow$ $b_1^2 = a_1 = 1$

$b_2 - 1 = 0$ $\Rightarrow$ $b_2^2 = a_2 - 1 = 1$ $\Rightarrow$ $a_2 = 2$

$b_3 - 1 = 0$ $\Rightarrow$ $b_3^2 = a_3 - 2 = 1$ $\Rightarrow$ $a_2 = 3$ and soon

hence $a_n = n$ $\therefore$ $\sum_{i=1}^{100} a_1 = 1 + 2 + 3 + ... + 100 = 5050$

19. $\sum_{i=1}^{n-1} (a_i + 3a_{i+1})^2 \le 0 \Rightarrow \frac{a_{i+1}}{a_i} = -\frac{-1}{3}$ G.P. $8, \frac{-8}{3}, \frac{8}{9}, \frac{-8}{27}, \frac{8}{81}$

Sum $8\left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81}\right) \Rightarrow \frac{8}{81}$ $(81 - 27 + 9 - 3 + 1) = \frac{488}{81}$

20. $a_n = (x)^{\frac{1}{2^n}} + (y)^{\frac{1}{2^n}}$; $b_n = (x)^{\frac{1}{2^n}} - (y)^{\frac{1}{2^n}}$, $n \in N$; $a_1 . a_2 . a_3$ ...... a_n

$$= (x^{1/2} + y^{1/2})\left(x^{\frac{1}{2^2}} + y^{\frac{1}{2^2}}\right) .... \left(x^{\frac{1}{2^n}} + y^{\frac{1}{2^n}}\right) \times \frac{(x^{\frac{1}{2^n}} - y^{\frac{1}{2^n}})}{b_n} = \frac{x-y}{b_n}$$

21. $2a_{i+1} = a_i + a_{i+2}$

$$\sum_{i=1}^{10} \frac{a_i \quad a_{i+1} \quad a_{i+2}}{2a_{i+1}} = \frac{1}{2}\sum_{i=1}^{10} a_i a_{i+2} = \frac{1}{2}\sum_{i=1}^{10} i(i+2) = \frac{1}{2}\left[\frac{10 \times 11 \times 21}{6} + \frac{2 \times 10(11)}{2}\right] = \frac{1}{2}[385 + 110] = \frac{495}{2}$$

ss

22. $\frac{n}{1.2.3} + \frac{n-1}{2.3.4} + ... + \frac{1}{n(n+1)(n+2)}$; $S_n = \sum_{r=1}^{n} \frac{n+1}{r(r+1)(r+2)} - \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$

Now $\sum_{r=1}^{n} \frac{1}{r \ (r+1)(r+2)} = \sum_{r=1}^{n} \frac{1}{2 \ (r+1)} \left[\frac{1}{r} - \frac{1}{r+2}\right] = \frac{1}{2}\left[\sum_{r=1}^{n} \frac{1}{r(r+1)} - \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}\right]$

$$= \frac{1}{2}\left[\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)\right] = \frac{1}{2}\left[\left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{n}{n+1} - \frac{n}{2(n+2)}\right] = \frac{n}{2}\left[\frac{1}{n+1} - \frac{1}{2(n+2)}\right]$$

Now $S_n = (n+1)\ \frac{n}{2}\left[\frac{1}{n+1} - \frac{1}{2(n+2)}\right] - \left(\frac{1}{2} - \frac{1}{n+2}\right)$

$$= \frac{n}{2} - \frac{n(n+1)}{4(n+2)} - \frac{n}{2(n+2)} = \frac{2n^2 + 4n - n^2 - n - 2n}{4(n+2)} = \frac{n^2 + n}{4(n+2)} = \frac{n(n+1)}{4(n+2)}$$

23. $T_n = \frac{3^n \times 5^n}{(5^n - 3^n)(5^{n+1} - 3^{n+1})}$; $T_n = \frac{1}{2}\left(\frac{3^n}{5^n - 3^n} - \frac{3^{n+1}}{5^{n+1} - 3^{n+1}}\right)$

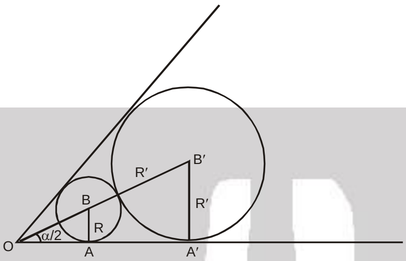
$T_1 = \frac{1}{2}\left(\frac{3}{5-3} - \frac{3^2}{5^2 - 3^2}\right)$; $T_2 = \frac{1}{2}\left(\frac{3^2}{5^2 - 3^2} - \frac{3^3}{5^3 - 3^3}\right)$

⋮ ⋮ ⋮

$$T_n = \frac{1}{2}\left(\frac{3^n}{5^n - 3^n} - \frac{3^{n+1}}{5^{n+1} - 3^{n+1}}\right)$$

$$S_n = \frac{1}{2}\left[\frac{3}{2} - \frac{3^{n+1}}{5^{n+1} - 3^{n+1}}\right] ; S_\infty = \frac{3}{4}$$

24. Radius of first circle = R Let Radius of second circle be = R'



$\frac{OB}{BA} = \text{cosec}\ \frac{\alpha}{2}$ $\quad\therefore\quad$ $OB = R\ \text{cosec}\frac{\alpha}{2}$

Now $\frac{OB'}{A'\ B'} = \text{cosec}\ \frac{\alpha}{2}$ $\quad\Rightarrow\quad$ $OB' = R'\ \text{cosec}\ \frac{\alpha}{2} = OB + R + R'$

$\Rightarrow R + R\ \text{cosec}\ \frac{\alpha}{2} + R' = R'\ \text{cosec}\frac{\alpha}{2} \Rightarrow R(1 + \text{cosec}\ \alpha/2) = R'\ (\text{cosec}\ \alpha/2 - 1) \Rightarrow\ R' = R\left(\frac{1+\sin\alpha/2}{1-\sin\alpha/2}\right)$

if radius of the third circle be R'', then similarly $R'' = R\left(\frac{1+\sin\alpha/2}{1-\sin\alpha/2}\right)^2$

So R + R' + R'' + ...n terms = $R\left[\frac{\left(\frac{1+\sin\ \ \alpha/2}{1-\sin\alpha/2}\right)^n - 1}{\left(\frac{1+\sin\alpha/2}{1-\sin\alpha/2}\right) - 1}\right] = R\left(\frac{1-\sin\alpha/2}{2\sin\alpha/2}\right)\left[\left(\frac{1+\sin\alpha/2}{1-\sin\alpha/2}\right)^n - 1\right]$

25. Given $(abc)^{2/3} = \frac{(a+b+c)}{3} \cdot \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow (ab + bc + ac)^3 = abc\ (a + b + c)^3$ .....(i)

Now consider the polynomial $p(x) = x^3 + mx^2 + nx + p$, with roots a, b, c, then have a + b + c = –m;ab + bc + ca = n ; abc = – p using these values equation (i) becomes $n^3 = m^3p$ ......(ii)

Hence, if $m \neq 0$, then equation p(x) = 0 can be written as $x^3 + mx^2 + nx + \frac{n^3}{m^3} = 0$

or $m^3x^3 + m^4 x^2 + nm^3x + n^3 = 0 \Rightarrow (mx + n)\ (m^2 x^2 + (m^3 - mn)\ x + n^2) = 0$

It follows that one of the roots of p(x) = 0 is $x_1 = -\frac{n}{m}$ and other two satisfy the condition $x_2x_3 = \frac{n^2}{m^2}$

$\Rightarrow \quad x_1^2 = x_2\ x_3$

Thus the roots are the terms of a geometric sequence. If should be noted that m, n ≠ 0 as in this case $x^3 + p = 0$ cannot have three real roots.

26. $t_n = S_n - S_{n-1} = \frac{n}{6}(2n^2 + 9n + 13) - \frac{(n-1)}{6}\{2(n-1)^2 + 9(n-1) + 13\}$

$= \frac{1}{6}[2n^3 + 9n^2 + 13n - 2(n-1)^3 - 9\ (n-1)^2 - 13(n-1)] = (n + 1)^2\ \sum_{r=1}^{\infty} \frac{1}{r\ .\ (r+1)} = \sum_{r=1}^{\infty}\left(\frac{1}{r} - \frac{1}{r+1}\right) = 1$

27. $\alpha + \beta = \frac{-\ b}{a}$, $\alpha\beta = \frac{c}{a} \Rightarrow (\alpha + \beta),\ \alpha^2 + \beta^2,\ \ \alpha^3 + \beta^3$ are in G.P then $(\alpha^2 + \beta^2)^2 = (\alpha + \beta)\ (\alpha^3 + \beta^3)$

$\Rightarrow 2\alpha^2\beta^2 = \alpha\beta^3 + \beta\alpha^3 \Rightarrow \alpha\beta\ [\alpha^2 + \beta^2 - 2\alpha\ \beta] = 0$ so $\alpha\ \beta\ (\alpha - \beta)^2 = 0 \Rightarrow \frac{c}{a} . \frac{b^2 - 4ac}{a^2} = 0,\ a \neq 0 \Rightarrow\ c\cdot\ \Delta = 0$

28. $S_n = (1 + 2T_n)(1 - T_n)$ $\Rightarrow$ $T_1 = (1 + 2T_1)(1 - T_1)$

$T_1 = 1 - T_1 + 2T_1 - 2T_1^2$ $\Rightarrow$ $2T_1^2 = 1$ $\Rightarrow$ $T_1 = \frac{1}{\sqrt{2}}$

$S_2 = T_1 + T_2 = (1 + 2T_2)(1 - T_2)$ $\Rightarrow$ $T_1 + T_2 = 1 - T_2 + 2T_2 - 2T_2^2$

$T_1 = 1 - 2T_2^2$ $\Rightarrow$ $2T_2^2 = 1 - \frac{1}{\sqrt{2}}$

$T_2^2 = \frac{\sqrt{2}-1}{2\sqrt{2}}$ $\Rightarrow$ $T_2^2 = \frac{2-\sqrt{2}}{4}$ $\Rightarrow$ a = 4, b = 2 $\Rightarrow$ a + b = 6